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Synthetic Theory of Superconnections

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In our preceding two papers we have developed synthetic differential supergeometry up to the basic theory of differential forms. In this paper we give the notion of connection, as well as its accompanying notions of connection form and curvature form, superized in our synthetic context, and establish the second Bianchi identity synthetically.

INTRODUCTION

The theory of elementary particles has been getting more and more geometric. The intimate relations between Yang–Mills theories in physics and the theory of connections in mathematics are widely known in both the mathematics and physics communities. The differential geometric foundations of gauge theories are firmly established.

Each elementary particle has to abide by one of the two kinds of statistics, namely, Bose–Einstein statistics or Fermi–Dirac statistics. Particles subject to the former statistics are called bosons, while those subject to the latter statistics are called fermions. Supergeometry has enabled mathematical physicists to deal with both kinds of elementary particles on an equal footing, providing the theory of connections with a super flavor.

Synthetic differential geometry is a vanguard of modern differential geometry, in which infinitesimals are abundantly and coherently available. To synthetic differential geometers the word "infinitesimal" is no longer a jaw breaker, but a magic wand. In spite of many mathematicians' studied indifference to infinitesimals per se, synthetic differential geometers are well aware that microlinear spaces, which are spaces infinitesimally indistinguishable from Euclidean spaces, are to replace smooth manifolds, just as Riemann integrals have been replaced completely by Lebesgue integrals. In the mathe-

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matics of our age, Riemann integrals are only a halfway house in memory of Riemann, one of the greatest mathematicians of the 19th century. It is doubtful that, without the glory of his name, they would have survived to the concluding decade of the 20th century.

The exact formulation of supergeometry, in particular, the exact formulation of supermanifold, is not as easy as was first thought. Our main platform is that the superization of microlinear space gives the right direction to follow in supergeometry. We have already developed synthetic differential supergeometry up to the basic theory of differential forms (Nishimura (1998, 1999). The principal objective of this paper is to develop the synthetic theory of superconnection. The succeeding section is devoted to somewhat lengthy preliminaries. In Section 2 we superize the notion of connection in our synthetic context, and discuss superconnection form (gauge potential) and covariant exterior differentiation. In Section 4 we introduce two kinds of curvature form (gauge field) and compare them. Only one of them is to satisfy the so-called second Bianchi identity. Superconnection forms and curvature forms of induced superconnections are discussed in Section 3 and the concluding part of Section 4.

As is usual in synthetic differential geometry, the reader should presume throughout the paper that we are working in a (not necessarily Boolean) topos, so that the excluded middle and Zorn's lemma have to be avoided. Objects of the topos go under such aliases as a "space," a "set," etc.

1. PRELIMINARIES

1.1. Basic Superalgebra

Let $\mathbb{Z}$ denote the set of integers, whose elements are usually written i, j, k, . . . , with or without subscripts. Let $\mathbb{Z}_2$ denote the set of integers mod 2, whose elements are usually written $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, . . . , with or without subscripts. We usually denote 0 mod 2 by $\mathbf{0}$ and 1 mod 2 by $\mathbf{1}$, though integers are sometimes regarded as elements of $\mathbb{Z}_2$. For any $\mathbf{p} \in \mathbb{Z}_2$, $(-1)^{\mathbf{p}}$ denotes 1 or -1 as $\mathbf{p} = \mathbf{0}$ or $\mathbf{p} = \mathbf{1}$. Both $\mathbb{Z}$ and $\mathbb{Z}_2$ are commutative rings in standard sense. A *superring* is a $\mathbb{Z}_2$-graded ring. Given a superring $\mathcal{S}$, we will often write $\mathcal{S}^{\mathbf{0}}$ or $\mathcal{S}_{\mathbf{e}}$ for its *even part* and $\mathcal{S}^{\mathbf{1}}$ or $\mathcal{S}_{\mathbf{o}}$ for its *odd part*. We say that $\mathcal{S}$ is *graded commutative* if for any $a \in \mathcal{S}^{\mathbf{p}}$ and any $b \in \mathcal{S}^{\mathbf{q}}$ we have

(1.1) $ab = (-1)^{\mathbf{pq}}ba.$

Now we choose, once and for all, a graded commutative superring $\mathbb{R}$ intended to play the role of real numbers in our supermathematics. So we have the following axiom:

(1.2) $\mathbb{R}$ is a graded commutative superring.

A *left* $\mathbb{R}$-*supermodule* is a left $\mathbb{R}$-module $\mathcal{M}$ whose underlying Abelian group is decomposed into even and odd parts $\mathcal{M}_{\mathbf{e}}$ and $\mathcal{M}_{\mathbf{o}}$ (also written $\mathcal{M}^{\mathbf{0}}$ and $\mathcal{M}^{\mathbf{1}}$), respectively, such that

(1.3) If $a \in \mathbb{R}^{\mathbf{p}}$ and $u \in \mathcal{M}^{\mathbf{q}}$, then $au \in \mathcal{M}^{\mathbf{p}+\mathbf{q}}$.

The notion of a *right* $\mathbb{R}$-*supermodule* is defined similarly. It is a truism that $\mathbb{R}$ can canonically be regarded as both left- and right $\mathbb{R}$-supermodules. It is well known that every left $\mathbb{R}$-supermodule $\mathcal{M}$ can be regarded as a right $\mathbb{R}$-supermodule in the sense that for any $a \in \mathbb{R}^{\mathbf{p}}$ and any $u \in \mathcal{M}^{\mathbf{q}}$, we have

(1.4) $ua = (-1)^{\mathbf{pq}}au$.

By the same token, every right $\mathbb{R}$-supermodule can be regarded as a left $\mathbb{R}$-supermodule, so that we can feel free to use the term “$\mathbb{R}$-supermodule” without an adjective “left” or “right.” In addition, any $\mathbb{R}$-supermodule $\mathcal{M}$ is an $\mathbb{R}$-bimodule in the sense that for any $u \in \mathcal{M}$ and any $a, b \in \mathbb{R}$ we have

(1.5) $(au)b = a(ub)$.

Each element u of an $\mathbb{R}$-supermodule $\mathcal{M}$ is decomposed uniquely into its even and odd parts $u_{\mathbf{e}}$ and $u_{\mathbf{o}}$, so that $u = u_{\mathbf{e}} + u_{\mathbf{o}}$ with $u_{\mathbf{e}} \in \mathcal{M}_{\mathbf{e}}$ and $u_{\mathbf{o}} \in \mathcal{M}_{\mathbf{o}}$. If u is even or odd, then it is called *pure*, in which $|u|$ is defined to be $\mathbf{0}$ or $\mathbf{1}$ according as $u \in \mathcal{M}_{\mathbf{e}}$ or $u \in \mathcal{M}_{\mathbf{o}}$.

Given $\mathbb{R}$-supermodules $\mathcal{M}$ and $\mathcal{N}$, an $\mathbb{R}$-homomorphism φ from the right $\mathbb{R}$-module $\mathcal{M}$ to the right $\mathbb{R}$-module $\mathcal{N}$ is called *even* or *odd* according as, for any $\mathbf{p} \in \mathbb{Z}_2$ and any $u \in \mathcal{M}^{\mathbf{p}}$, we have

(1.6) $f(u) \in \mathcal{N}^{\mathbf{p}}$, or
(1.7) $f(u) \in \mathcal{N}^{\mathbf{p}+\mathbf{1}}$.

The additive group of even or odd $\mathbb{R}$-homomorphisms from $\mathcal{M}$ to $\mathcal{N}$ is denoted by $\mathrm{Hom}_{\mathbf{e}}(\mathcal{M}, \mathcal{N})$ or $\mathrm{Hom}_{\mathbf{o}}(\mathcal{M}, \mathcal{N})$ [also written $\mathrm{Hom}^{\mathbf{0}}(\mathcal{M}, \mathcal{N})$ or $\mathrm{Hom}^{\mathbf{1}}(\mathcal{M}, \mathcal{N})$]. We write $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ for their direct sum $\mathrm{Hom}_{\mathbf{e}}(\mathcal{M}, \mathcal{N}) \oplus \mathrm{Hom}_{\mathbf{o}}(\mathcal{M}, \mathcal{N})$, which can be considered as an $\mathbb{R}$-supermodule in the sense that for any $a \in \mathbb{R}$, any $u \in \mathcal{M}$, and any $f \in \mathrm{Hom}(\mathcal{M}, \mathcal{N})$ we have

(1.8) $(af)(u) = af(u)$.

An $\mathbb{R}$-*superalgebra* is an $\mathbb{R}$-algebra $\mathcal{A}$ which is a superring and an $\mathbb{R}$-supermodule with respect to the same $\mathbb{Z}_2$-grading such that for any $u, v \in \mathcal{A}$ and any $a, b \in \mathbb{R}$ we have

(1.9) $(a\mathrm{u})(vb) = a(uv)b$.

An example of an $\mathbb{R}$-superalgebra is the totality of $\mathbb{R}$-valued functions on a set with componentwise operations, in which its even and odd elements are $\mathbb{R}_{\mathbf{e}}$-valued and $\mathbb{R}_{\mathbf{o}}$-valued ones. A *homomorphism of* $\mathbb{R}$-*superalgebras* is a homomorphism of their underlying $\mathbb{R}$-algebras preserving $\mathbb{Z}_2$-gradings. Given two $\mathbb{R}$-superalgebras $\mathcal{A}$ and $\mathcal{B}$, we will often write $\mathrm{Spec}_{\mathcal{B}}\mathcal{A}$ for the set of homomorphisms of $\mathbb{R}$-superalgebras from $\mathcal{A}$ to $\mathcal{B}$.

The *polynomial* $\mathbb{R}$*-superalgebra* $\mathbb{R}[X_1, \ldots, X_n]$ of *variables* $X_1, \ldots, X_n$ with each of the variables being named as either even or odd is the graded commutative $\mathbb{R}$-superalgebra freely generated by $X_1, \ldots, X_n$ over $\mathbb{R}$. It is characterized by the following universal property (Manin, 1988, Chapter 3, §2, Item 5).

Proposition 1.1. For any graded commutative $\mathbb{R}$-superalgebra $\mathcal{A}$ and any pure elements $a_1, \ldots, a_n$ of $\mathcal{A}$ with $|a_i|, = |X_i|$ $(1 \leq i \leq n)$, there exists a unique homomorphism φ of $\mathbb{R}$-superalgebras from $\mathbb{R}[X_1, \ldots, X_n]$ to $\mathcal{A}$ such that $\varphi(X_i) = a_i$ $(1 \leq i \leq n)$.

An ideal $\mathcal{I}$ of an $\mathbb{R}$-superalgebra $\mathcal{A}$ is called a *superideal* of $\mathcal{A}$ if both the even and odd parts of each element of $\mathcal{I}$ belong to $\mathcal{I}$.

1.2. Weil Superalgebras and Supermicrolinearity

A *Weil superalgebra* is a graded commutative $\mathbb{R}$-superalgebra $\mathfrak{W}$ which, regarded as an $\mathbb{R}$-module, is to be written as $\mathfrak{W} = \mathbb{R} \oplus \mathfrak{m}$ with the first component being the $\mathbb{R}$-superalgebra structure and the second being a finite-dimensional nilpotent superideal (called the *superideal of augmentation*). By way of example, the quotient superalgebra of the polynomial superalgebra $\mathbb{R}[X_1, \ldots, X_n]$ with respect to the superideal generated by $\{X_i X_j \mid 1 \leq i \leq n\}$ is a Weil superalgebra and is denoted by $\mathfrak{W}(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ with $\mathbf{p}_i = |X_i|$ $(1 \leq i \leq n)$. Given Weil superalgebras $\mathfrak{W}_1$ and $\mathfrak{W}_2$ with their superideals of augmentation $\mathfrak{m}_1$ and $\mathfrak{m}_2$, respectively, a homomorphism of $\mathbb{R}$-superalgebras $\varphi\colon \mathfrak{W}_1 \to \mathfrak{W}_2$ is said to be a *homomorphism of Weil superalgebras* if it preserves their superideals of augmentation, i.e., if $\varphi(\mathfrak{m}_1) \subset \mathfrak{m}_2$. A finite limit diagram of $\mathbb{R}$-superalgebras is said to be a *good finite limit diagram of Weil superalgebras* if every object occurring in the diagram is a Weil superalgebra and every morphism occurring in the diagram is a homomorphism of Weil superalgebras. The diagram obtained from a good finite limit diagram of Weil superalgebras by taking $\mathrm{Spec}_{\mathbb{R}}$ is called a *quasi-colimit diagram of supersmall objects*.

The super version of the general Kock axiom, called the *general super-Kock axiom*, goes as follows:

(1.10) For any Weil superalgebra $\mathfrak{W}$, the canonical $\mathbb{R}$-superalgebra homomorphism $\mathfrak{W} \to \mathbb{R}^{\mathrm{Spec}_{\mathbb{R}}(\mathfrak{W})}$ is an isomorphism.

Spaces of the form $\mathrm{Spec}_{\mathbb{R}}(\mathfrak{W})$ for some Weil superalgebras $\mathfrak{W}$ are called *superinfinitesimal spaces* or *supersmall objects*. The superinfinitesimal space corresponding to the Weil superalgebra $\mathfrak{W}(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ is denoted by $D(\mathbf{p}_1, \ldots, \mathbf{p}_n)$. In particular, $D(\cdot)$, $D(\mathbf{0})$, and $D(\mathbf{1})$ are denoted also by 1, D, and $\overline{D}$, respectively. As an example, by Proposition 1.1, D, $\overline{D}$, and $D(\mathbf{0}, \mathbf{1})$ are to be identified with $\{d \in \mathbb{R}_{\mathbf{e}} | d^2 = 0\}$. $\{d \in \mathbb{R}_{\mathbf{o}} | d^2 = 0\}$, and $\{(d_1, d_2) \in \mathbb{R}_{\mathbf{e}} \times \mathbb{R}_{\mathbf{o}} | d_1^2 = d_2^2 = d_1 d_2\}$, respectively. It is easy but interesting to see that $\overline{D} = \mathbb{R}_{\mathbf{o}}$, from which and the general super-Kock axiom it follows that every function from $\mathbb{R}_{\mathbf{o}}$ to $\mathbb{R}$ is linear (Dewitt, 1984, Exercise 1.1). Given $\mathbf{p} \in \mathbb{Z}_2$, $D^{\mathbf{p}}$ denotes D or $\overline{D}$ according as $\mathbf{p}$ is $\mathbf{0}$ or $\mathbf{1}$.

The superinfinitesimal space $D(\mathbf{0}, \mathbf{1})$ plays a very important role in our discussion of tangency. First we note that $D(\mathbf{0}, \mathbf{1})$ can be identified with the subset of $\mathbb{R}$ consisting of all $\mathbb{d} \in \mathbb{R}$ such that $\mathbb{d}_{\mathbf{e}}^2 = \mathbb{d}_{\mathbf{o}}^2 = \mathbb{d}_{\mathbf{e}}\mathbb{d}_{\mathbf{o}} = 0$. Under this identification $(d_1, d_2) \in D(\mathbf{0}, \mathbf{1})$ corresponds to $d_1 + d_2 \in \mathbb{R}$. What concerns us most about $D(\mathbf{0}, \mathbf{1})$ is that the space $D(\mathbf{0}, \mathbf{1})$, regarded as a subset of $\mathbb{R}$, is closed under the left and right actions of $\mathbb{R}$ on itself, while D and $\overline{D}$ are not. More specifically, given $a \in \mathbb{R}$ and $(d_1, d_2) \in D(\mathbf{0}, \mathbf{1})$, $a(d_1, d_2)$ and $(d_1, d_2)\,a$ go as follows:

(1.11) $a(d_1, d_2) = a_{\mathbf{e}}d_1 + a_{\mathbf{o}}d_2, a_{\mathbf{o}}d_1 + a_{\mathbf{e}}d_2)$
(1.12) $(d_1, d_2)a = (d_1 a_{\mathbf{e}} + d_2 a_{\mathbf{o}}, d_1 a_{\mathbf{o}} + d_2 a_{\mathbf{e}})$

Just as the general Kock axiom paved the way for the introduction of microlinear spaces, its super version invokes the notion of a *supermicrolinear space*, which is by definition a space M abiding by the following condition:

(1.13) For any good finite limit diagram of Weil superalgebras with its limit W, the diagram obtained by taking $\mathrm{Spec}_{\mathbb{R}}$ and then exponentiating over M is a limit diagram with its limit $M^{\mathrm{Spec}_{\mathbb{R}} W}$.

The following proposition guarantees that we have plenty of supermicrolinear spaces.

Proposition 1.2. (1) $\mathbb{R}_{\mathbf{e}}$ and $\mathbb{R}_{\mathbf{o}}$ be supermicrolinear spaces.

(2) The class of supermicrolinear spaces is closed under limits and exponentiation by an arbitrary space.

1.3. Differential Calculus

The super version of the Kock-Lawvere axiom, which is subsumed under the super version of the general Kock axiom discussed in the previous subsection, goes as follows:

(1.14) For any function $f: D \to \mathbb{R}$, there exists a unique $b \in \mathbb{R}$ such that $f(d) = f(0) + bd$ for any $d \in D$.

(1.15) For any function $g: \overline{D} \to \mathbb{R}$, there exists a unique $c \in \mathbb{R}$ such that $g(d) = g(0) + cd$ for any $d \in \overline{D}$.

The unique b and c in the above axioms are usually denoted by $(f\overleftarrow{\mathbf{D}}_{\mathbf{0}})(0)$, and $(f\overleftarrow{\mathbf{D}}_{\mathbf{1}})(0)$, respectively. The axioms (1.14) and (1.15) are equivalent to the following two axioms:

(1.16) For any function $f: D \to \mathbb{R}$, there exists a unique $b' \in \mathbb{R}$ such that $f(d) = f(0) + db'$ for any $d \in D$.

(1.17) For any function $g: \overline{D} \to \mathbb{R}$, there exists a unique $c' \in \mathbb{R}$ such that $g(d) = g(0) + dc'$ for any $d \in \overline{D}$.

The unique b' and c' in the above axioms are usually denoted by $({}_{\mathbf{0}}\overrightarrow{\mathbf{D}}f)(0)$ and $({}_{\mathbf{0}}\overleftarrow{\mathbf{D}}f)(0)$, respectively. These four axioms as a whole are called the *super-Kock–Lawvere axiom*. For details of elementary differential calculus in this direction the reader is referred to Nishimura (1998, §3).

We conclude this subsection by a definition. An $\mathbb{R}$-supermodule $\mathcal{M}$ is said to be *graded Euclidean* if it abides by the following conditions:

(1.18) For any function $f: D \to \mathcal{M}$, there exists a unique $x \in \mathcal{M}$ such that $f(d) = f(0) + xd$ for any $d \in D$.

(1.19) For any function $g: \overline{D} \to \mathcal{M}$, there exists a unique $y \in \mathcal{M}$ such that $g(d) = g(0) + yd$ for any $d \in \overline{D}$.

1.4. Supermicrocubes

A supermicrolinear space M shall be chosen arbitrarily once and for all. Given $(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in (\mathbb{Z}_2)^n$, a *pure n-supermicrocube of type* $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ *on M* is a function from $D\mathbf{p}_1 \times \ldots \times D\mathbf{p}_n$ to M. We denote by $\mathbf{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_n}M$ the totality of pure n-supermicrocubes of type $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ on M. We denote by $\mathbf{T}^{\mathbf{p}}M$ the set-theoretic union of $\mathbf{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_n}M$ for all $(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in (\mathbb{Z}_2)^n$. In particular, $\mathbf{T}^1M$ is usually denoted by $\mathbf{T}M$, and their elements are called *pure supervectors tangent to M*. Given $\gamma \in \mathbf{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_n}M$ and $e \in D^{\mathbf{p}_i}$, γ^i_e denotes the mapping

$$(d_1, \ldots, d_{n-1}) \in D^{\mathbf{p}_1} \times \ldots \times D^{\mathbf{p}_{i-1}} \times D^{\mathbf{p}_{i+1}} \times \ldots \times D^{\mathbf{p}_n}$$
$$\mapsto \gamma(d_1, \ldots, d_{i-1}, e, d_{i+1}, \ldots, d_{n-1})$$

which is surely a pure $(n-1)$-supermicrocube of type $(\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n)$.

An *n-supermicrocube on M* is a mapping from $D(\mathbf{0}, \mathbf{1})^n$ to M. We denote by $\mathfrak{I}^nM$ the totality of n-supermicrocubes on M. In particular, $\mathfrak{I}^1M$ is usually denoted by $\mathfrak{I}M$, and their elements are called *supervectors tangent to M*.

Given $x \in M$, we denote the sets $\{t \in \mathbf{T}^{\mathbf{0}}M | t(0) = x\}$, $\{t \in \mathbf{T}^{\mathbf{1}}M | t(0) = x\}$, and $\{\mathfrak{t} \in \mathfrak{I}M | \mathfrak{t}(0) = x\}$ by $\mathbf{T}^0_x M$, $\mathbf{T}^1_x M$, and $\mathfrak{I}_x M$, respectively. We have shown (Nishimura, 1998, §4) that $\mathfrak{I}_x M$ is an $\mathbb{R}$-supermodule and that its even and odd parts can naturally be identified with $\mathbf{T}^0_x M$ and $\mathbf{T}^1_x M$. We have noted there also that the $\mathbb{R}$-supermodule $\mathfrak{I}_x M$ is graded Euclidean.

Given $(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in (\mathbb{Z}_2)^n$, the canonical injection of $D^{\mathbf{p}_1} \times \ldots \times D^{\mathbf{p}_n}$ into $D(\mathbf{0}, \mathbf{1})^n$ and the canonical projection of $D(\mathbf{0}, \mathbf{1})^n$ onto $D^{\mathbf{p}_1} \times \ldots \times D^{\mathbf{p}_n}$ are denoted by $\iota_{\mathbf{p}_1,\ldots,\mathbf{p}_n}$ and $\pi_{\mathbf{p}_1,\ldots,\mathbf{p}_n}$, respectively. The totality of $\overline{\gamma} \in \mathfrak{I}^n M$ with $\overline{\gamma} \circ \iota_{\mathbf{p}_1,\ldots,\mathbf{p}_n} \circ \pi_{\mathbf{p}_1,\ldots,\mathbf{p}_n} = \overline{\gamma}$ can and shall hereafter be identified with $T^{\mathbf{p}_1,\ldots,\mathbf{p}_n}M$.

1.5. Exterior Differential Calculus

Given $\gamma \in \mathbf{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_n}M$ and $a \in \mathbb{R}^{\mathbf{q}}$, pure n-supermicrocubes $\gamma \underset{i}{\cdot} a$ and $a \underset{i}{\cdot} \gamma$ of type $(\mathbf{p}_1, \ldots, \mathbf{p}_i + \mathbf{q}, \ldots, \mathbf{p}_n)$ on M $(1 \leq i \leq n)$ are defined by

(1.20) $(\gamma \underset{i}{\cdot} a)(d_1, \ldots, d_n) = \gamma(d_1, \ldots, ad_i, \ldots, d_n)$

(1.21) $(a \underset{i}{\cdot} \gamma)(d_1, \ldots, d_n) = \gamma(d_1, \ldots, d_i a, \ldots, d_n)$

for any $(d_1, \ldots, d_n) \in D^{\mathbf{p}_1} \times \ldots \times D^{\mathbf{p}_i + \mathbf{q}} \times \ldots \times D^{\mathbf{p}_n}$.

Given $\gamma \in T^{\mathbf{p}_1,\ldots,\mathbf{p}_n}M$ and $\sigma \in \mathfrak{Perm}_n$, a pure n-supermicrocube $\Sigma_\sigma(\gamma)$ of type $(\mathbf{p}_{\sigma^{-1}(1)}, \ldots, \mathbf{p}_{\sigma^{-1}(n)})$ on M is defined as follows:

(1.22) $\Sigma_\sigma(\gamma)(d_1, \ldots, d_n) = \gamma(d_{\sigma(1)}, \ldots, d_{\sigma(n)})$ for any $(d_1, \ldots, d_n) \in D^{\mathbf{p}_{\sigma^{-1}(1)}} \times \ldots \times D^{\mathbf{p}_{\sigma^{-1}(n)}}$.

A *graded differential n-form on M* is a mapping θ from $\mathbf{T}^n M$ to $\mathbb{R}$ abiding by the following conditions:

(1.23) $\theta(\gamma \underset{i}{\cdot} a) = \theta(a \underset{i+1}{\cdot} \gamma)$ $(1 \leq i \leq n-1)$, while $\theta(\gamma \underset{n}{\cdot} a) = \theta(\gamma)a$ for any $\mathbf{q} \in \mathbb{Z}_2$, any $a \in \mathbb{R}^{\mathbf{q}}$, and any $\gamma \in \mathbf{T}^n M$.

(1.24) If γ is a pure n-microsquare of type $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ on M, then $\theta(\Sigma_{(i,j)}\gamma) = (-1)^{1+\eta_{i,j}}\theta(\gamma)$ $(1 \leq i < j \leq n)$, where $\eta_{i,j} = \mathbf{p}_i \Sigma^j_{h=i+1}\mathbf{p}_h + \mathbf{p}_j \Sigma^{j-1}_{h=i+1}\mathbf{p}_h$.

We denote by $\Xi_n(M)$ the totality of graded differential n-forms on M.

Given $\overline{\gamma} \in \mathfrak{T}^n M$ and $a \in \mathbb{R}$, n-supermicrocubes $\overline{\gamma} \underset{i}{\cdot} a$ and $a \underset{i}{\cdot} \overline{\gamma}$ on M $(1 \leq i \leq n)$ are defined as in (1.18) and (1.19), respectively. Given $\overline{\gamma} \in \mathfrak{T}^n M$ and $\sigma \in \mathfrak{Perm}_n$, an n-supermicrocube $\Sigma_\sigma(\overline{\gamma})$ on M is defined as in (1.20). A *differential n-form on M* is a mapping $\overline{\theta}$ from $\mathfrak{T}^n M$ to $\mathbb{R}$ subject to the following conditions:

(1.25) $\overline{\theta}(\overline{\gamma} \underset{i}{\cdot} a) = \overline{\theta}(a \underset{i+1}{\cdot} \overline{\gamma})$ $(1 \leq i \leq n-1)$, while $\overline{\theta}(\overline{\gamma} \underset{n}{\cdot} a) = \overline{\theta}(\overline{\gamma})a$ for any $a \in \mathbb{R}$ and any $\overline{\gamma} \in \mathfrak{T}^n M$.

(1.26) If $\bar{\gamma}$ is a pure n-supermicrocube of type $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ on M, then $\bar{\theta}(\Sigma_{(i,j)}\bar{\gamma}) = (-1)^{1+\eta_{i,j}}(\bar{\gamma})$ $(1 \leq i < j \leq n)$, where $\eta_{i,j} = \mathbf{p}_i \sum_{h=i+1}^{j}\mathbf{p}_h + \mathbf{p}_j \sum_{h=i+1}^{j-1}\mathbf{p}_h$.

We denote by $\Xi_n(M)$ the totality of differential n-forms on M. We have shown (Nishimura, 1999, Proposition 1.2) that there is a natural bijective correspondence between $\underline{\Xi}_n(M)$ and $\Xi_n(M)$. We have shown (Nishimura (1999, Proposition 2.5) that, given $\bar{\theta} \in \Xi_n(M)$, there exists a unique $\mathbf{d}\bar{\theta} \in \Xi_{n+1}(M)$ such that for any $(\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}) \in (\mathbb{Z}_2)^{n+1}$, any $\gamma \in \mathbf{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_{n+1}}M$, and any $(e_1, \ldots, e_{n+1}) \in D_1^{\mathbf{p}} \times \ldots \times D_{n+1}^{\mathbf{p}}$, we have

(1.27) $d\bar{\theta}(\gamma)e_1 \ldots e_{n+1} = \sum_{i=1}^{n+1} (-1)^{j+\alpha_i}(\theta(\gamma_0^i) - \theta(\gamma_{e_i}^i))\, e_1 \ldots \hat{e}_i \ldots e_{n+1}$

with $\alpha_i = \mathbf{p}_i(\Sigma_{h\neq i}\mathbf{p}_h)$.

1.6. Supervector Bundles

A mapping ζ: $E \to M$ of supermicrolinear spaces is called a *supervector bundle* provided that $E_x = \zeta^{-1}(x)$ is a Euclidean $\mathbb{R}$-supermodule for any $x \in M$. We call M the *base space* of ζ and E_x the *fiber over x*. The totality of mappings λ: $M \to E$ with $\zeta \circ \lambda = \mathrm{id}_M$ (id_M denotes the identity transformation of M) is denoted by Sec ζ. The totality of $\bar{\mathfrak{t}} \in \mathfrak{T}E$ with $\zeta \circ \bar{\mathfrak{t}} = \mathbb{0}$ [the zero supervector tangent to M at $\zeta \circ \bar{\mathfrak{t}}(0)$] is to be considered as a supervector bundle over E and is to be denoted $\mathbf{V}(E)$.

The *tangent bundle* τ_M: $M^{D(\mathbf{0},\mathbf{1})} \to M$ is a supervector bundle, where τ_M assigns, to each $\mathfrak{t} \in M^{D(\mathbf{0},\mathbf{1})}$, $\mathfrak{t}(0) \in M$. If $\mathcal{N}$ is a Euclidean $\mathbb{R}$-supermodule which is supermicrolinear, then the trivial bundle $M \times \mathcal{N} \to M$ is a supervector bundle.

Various algebraic constructions in linear superalgebra can be carried over to supervector bundles. If ζ: $E \to M$ and η: $F \to M$ are supervector bundles over the same base space M, then their Whitney sum $\zeta \oplus \eta$ and the natural protection $\pi_{\mathscr{L}(\zeta,\eta)}$: $\mathscr{L}(\zeta,\eta) \to M$ are supervector bundles, where $\mathscr{L}(\zeta, \eta)$ denotes the set-theoretic union of Hom (ζ_x, η_x) for all $x \in M$.

If φ: $M \to N$ is a map of supermicrolinear spaces and η: $F \to N$ is a supervector bundle, then the notion of a graded differential n-form on M and that of a differential n-form on M discussed in the preceding section can be generalized easily to that of a *graded differential n-form on M with values in* η *relative to* φ and that of a *differential n-form on M with values in* η *relative to* φ . We denote by $\underline{\Xi}^n(M \xrightarrow{\varphi} N; \xi)$ and $\Xi^n(M \xrightarrow{\varphi} N; \xi)$ the totality of graded differential n-forms on M with values in η relative to φ and that of differential n-forms on M with values in η relative to φ, respectively. They are to be identified naturally as $\underline{\Xi}^n(M)$ and $\Xi^n(M)$. If $N = M$ and φ is the identity map id_M of M, then $\underline{\Xi}^n(M \xrightarrow{\varphi} N; \eta)$ and $\Xi^n(M \xrightarrow{\varphi} N; \eta)$ are denoted also by $\underline{\Xi}^n(M; \eta)$ and $\Xi^n(M; \eta)$, respectively. If η is furthermore a trivial bundle $M \times \mathbb{R} \to M$, then $\underline{\Xi}^n(M; \eta)$ and $\Xi^n(M; \eta)$ degenerate into $\underline{\Xi}^n(M)$ and $\Xi^n(M)$, respectively.

2. SUPERCONNECTIONS

Let $\zeta: E \to M$ be a supervector bundle. A *superconnection* on ζ is a mapping $\nabla: M^{D(\mathbf{0},\mathbf{1})} \times_M E \to E^{D(\mathbf{0},\mathbf{1})}$ such that for any $(\mathbb{t}, v) \in M^{D(\mathbf{0},\mathbf{1})} \times_M E$, any $a \in \mathbb{R}$, and any $\mathbb{d} \in D^{(\mathbf{0},\mathbf{1})}$ we have that

(2.1) $\nabla(\mathbb{t}, v)(0) = v$

(2.2) $\nabla(\mathbb{t}a, v)(\mathbb{d}) = \nabla(\mathbb{t}, v)(a\mathbb{d})$

(2.3) $\nabla(\mathbb{t}, va)(\mathbb{d}) = (\nabla(\mathbb{t}, v)(\mathbb{d}))a$

(2.4) The mapping $u \in E_{\mathbb{t}(0)} \mapsto \nabla(\mathbb{t}, u)(\mathbb{d}) \in E_{\mathbb{t}(\mathbb{d})}$, denoted by $p^{\nabla}_{(\mathbb{t},\mathbb{d})}$ or $p_{(\mathbb{t},\mathbb{d})}$, is bijective and preserves parities. Its inverse is denoted by $q^{\nabla}_{(\mathbb{t},\mathbb{d})} = q_{(\mathbb{t},\mathbb{d})}: E_{\mathbb{t}(\mathbb{d})} \to E_{\mathbb{t}(0)}$. We call $p_{(\mathbb{t},\mathbb{d})}$ the *parallel transport* from $\mathbb{t}(0)$ to $\mathbb{t}(\mathbb{d})$ along $\mathbb{t}$, while $q_{(\mathbb{t},\mathbb{d})}$ is called the *parallel transport* from $\mathbb{t}(\mathbb{d})$ to $\mathbb{t}(0)$ along $\mathbb{t}$.

If the supervector bundle $\zeta: E \to M$ is a trivial bundle $M \times \mathcal{N} \to M$, and if $\nabla(\mathbb{t}, \mathbb{t}(0), x))(\mathbb{d}) = (\mathbb{t}(\mathbb{d}), x)$ for any $\mathbb{t} \in M^{D(\mathbf{0},\mathbf{1})}$, any $x \in \mathcal{N}$, and any $\mathbb{d} \in D(\mathbf{0}, \mathbf{1})$, then the superconnection ∇ is called *trivial*.

Given $\bar{\mathbb{t}} \in E^{D(\mathbf{0},\mathbf{1})}$, we define $\overline{\omega}(\bar{\mathbb{t}}) \in E^{D(\mathbf{0},\mathbf{1})}$ to be

(2.5) $\overline{\omega}(\bar{\mathbb{t}}) = \bar{\mathbb{t}} - \nabla(\zeta \circ \bar{\mathbb{t}}, \bar{\mathbb{t}}(0))$

Since $\overline{\omega}(\bar{\mathbb{t}}) \in \mathbf{V}(E)$, there exist unique $\omega_{\mathbf{e}}(\bar{\mathbb{t}}), \omega_{\mathbf{o}}(\bar{\mathbb{t}}) \in E_{\zeta\circ\bar{\mathbb{t}}(0)}$ such that

(2.6) $\overline{\omega}(\bar{\mathbb{t}})(\mathbb{d}) = \bar{\mathbb{t}}(0) + \omega_{\mathbf{e}}(\bar{\mathbb{t}})\mathbb{d}_{\mathbf{e}} + \omega_{\mathbf{o}}(\bar{\mathbb{t}})\mathbb{d}_{\mathbf{o}}$

for any $\mathbb{d} \in D(\mathbf{0}, \mathbf{1})$. We define $\omega(\bar{\mathbb{t}})$ to be $\omega_{\mathbf{e}}(\bar{\mathbb{t}}) + \omega_{\mathbf{o}}(\bar{\mathbb{t}})$.

Proposition 2.1. Given $v \in E$ and $x \in M$ with $x = \zeta(v)$, the mapping $\bar{\mathbb{t}} \in (E^{D(\mathbf{0},\mathbf{1})})_v \mapsto \omega(\bar{\mathbb{t}}) \in E$ is homogeneous, so that ω is a differential 1-form on E with values in ζ relative to ζ.

Proof. By (2.2), $\overline{\omega}$ is homogeneous, so that for any $a \in \mathbb{R}$ and any $\mathbb{d} \in D(\mathbf{0}, \mathbf{1})$,

$$
\begin{aligned}
(2.7a)\quad & \overline{\omega}(\bar{\mathbb{t}}a)(\mathbb{d}) \\
&= \overline{\omega}(\bar{\mathbb{t}})(a\mathbb{d}) \\
&= \overline{\omega}(\bar{\mathbb{t}})((a_{\mathbf{e}}\mathbb{d}_{\mathbf{e}} + a_{\mathbf{o}}\mathbb{d}_{\mathbf{o}}) + (a_{\mathbf{e}}\mathbb{d}_{\mathbf{o}} + a_{\mathbf{o}}\mathbb{d}_{\mathbf{e}})) \\
&= \bar{\mathbb{t}}(0) + \omega_{\mathbf{e}}(\bar{\mathbb{t}})(a_{\mathbf{e}}\mathbb{d}_{\mathbf{e}} + a_{\mathbf{o}}\mathbb{d}_{\mathbf{o}}) + \omega_{\mathbf{o}}(\bar{\mathbb{t}})(a_{\mathbf{e}}\mathbb{d}_{\mathbf{o}} + a_{\mathbf{o}}\mathbb{d}_{\mathbf{e}}) \\
&= \bar{\mathbb{t}}(0) + (\omega_{\mathbf{e}}(\bar{\mathbb{t}})a_{\mathbf{e}} + \omega_{\mathbf{o}}(\bar{\mathbb{t}})a_{\mathbf{o}})\mathbb{d}_{\mathbf{e}} \\
&\quad + (\omega_{\mathbf{e}}(\bar{\mathbb{t}})a_{\mathbf{o}} + \omega_{\mathbf{o}}(\bar{\mathbb{t}})a_{\mathbf{e}})\mathbb{d}_{\mathbf{o}}
\end{aligned}
$$

Therefore

$$(2.7b)\quad \omega(\bar{\mathbb{t}}a) = (\omega_{\mathbf{e}}(\bar{\mathbb{t}})a_{\mathbf{e}} + \omega_{\mathbf{o}}(\bar{\mathbb{t}})a_{\mathbf{o}}) + (\omega_{\mathbf{e}}(\bar{\mathbb{t}})a_{\mathbf{o}} + \omega_{\mathbf{o}}(\bar{\mathbb{t}})a_{\mathbf{e}}) = \omega(\bar{\mathbb{t}})a$$

as was claimed. ■

We say that ω is the *superconnection form* of $\bigtriangledown$.

Proposition 2.2. For any $\mathbb{d} \in D(0, 1)$ and any $\bar{\mathbb{t}} \in E^{D(0,1)}$ we have

$$(2.8)\quad q_{(\zeta\circ\bar{\mathbb{t}},\mathbb{d})}(\bar{\mathbb{t}}(\mathbb{d})) = \bar{\mathbb{t}}(0) + \omega_{\mathbf{e}}(\bar{\mathbb{t}})\mathbb{d}_{\mathbf{e}} + \omega_{\mathbf{o}}(\bar{\mathbb{t}})\mathbb{d}_{\mathbf{o}}$$

Proof. Consider the mapping

$$(\mathbb{d}, \mathbb{d}') \in D(0, 1, 0, 1) \mapsto p_{(\zeta\circ\bar{\mathbb{t}},\mathbb{d})}(\bar{\mathbb{t}}(0) + \omega_{\mathbf{e}}(\bar{\mathbb{t}})\mathbb{d}'_{\mathbf{e}} + \omega_{\mathbf{o}}(\bar{\mathbb{t}})(\mathbb{d}'_{\mathbf{o}}) \in E$$

which coincides with $\bigtriangledown(\zeta \circ \bar{\mathbb{t}}, \bar{\mathbb{t}}(0))$ on the first axis and which coincides with $\overline{\omega}(\bar{\mathbb{t}})$ on the second axis. Therefore the mapping

$$\mathbb{d} \in D(0, 1) \mapsto p_{(\zeta\circ\bar{\mathbb{t}},\mathbb{d})}(\bar{\mathbb{t}}(0) + \omega_{\mathbf{e}}(\bar{\mathbb{t}})\mathbb{d}_{\mathbf{e}} + \omega_{\mathbf{o}}(\bar{\mathbb{t}})\mathbb{d}_{\mathbf{o}}) \in E$$

coincides with $\bar{\mathbb{t}}$, which implies the desired proposition. ■

Now suppose that we are given a mapping $\varphi : M \to N$ of microlinear spaces and a supervector bundle $\eta : F \to N$ endowed with a superconnection $\overline{\bigtriangledown}$, which shall be fixed throughout the rest of this section. Given a differential n-form θ on M with values in η relative to φ, we would like to define its *covariant exterior derivative* $\mathbf{D}_{\overline{\bigtriangledown}}\theta$, which is to be a differential $(n + 1)$-form on M with values in η relative to φ. It is not difficult to see that for any $(\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}) \in (\mathrm{Z}_2)^{n+1}$ and $\gamma \in \mathbf{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_{n+1}}M$, there exists a unique $\mathbf{D}_{\overline{\bigtriangledown}}\theta(\gamma) \in F_{\eta(\gamma(0,\ldots,0))}$ such that for any $(e_1, \ldots, e_{n+1}) \in D\mathbf{p}_1 \times \ldots \times D\mathbf{p}_{n+1}$ we have

$$(2.9)\quad \mathbf{D}_{\overline{\bigtriangledown}}\theta(\gamma)e_1 \ldots e_{n+1} = \sum_{i=1}^{n+1} (-1)^{i+\alpha_i}(\theta(\gamma_0^i - q^{\overline{\bigtriangledown}}_{(\varphi\circ\gamma_i,e_i)}(\theta(\gamma^i_{e_i})))e_i \ldots \hat{e}_i \ldots e_{n+1}$$

where γ_i is the tangent supervector to M assigning $\gamma(0, \ldots, 0, d, 0, \ldots, 0)$ (d is positioned at the ith slot) to each $d \in D^{\mathbf{p}_i}$ and $\alpha_i = \mathbf{p}_i(\Sigma_{n\neq i}\mathbf{p}_n)$. The crucial step in the proof that the mapping $\gamma \in \mathbf{T}^{n+1}M \mapsto \mathbf{D}_{\overline{\bigtriangledown}}\theta(\gamma)$ is indeed a graded differential $(n + 1)$-form on M with values in η relative to φ follows from the following two lemmas, as in Nishimura (1999, §2).

Lemma 2.3. We have

$$(2.10)\quad \mathbf{D}_{\overline{\bigtriangledown}}\theta(\gamma \underset{n+1}{\cdot} a) = \mathbf{D}_{\overline{\bigtriangledown}}\theta(\gamma)a$$

for any $(\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}) \in (\mathbb{Z}_2)^{n+1}$, any $\gamma \in \mathbf{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_{n+1}}M$, and any $a \in \mathbb{R}_{\mathbf{o}}$.

Proof. Let $(\underline{\mathbf{p}}_1, \ldots, \underline{\mathbf{p}}_n, \underline{\mathbf{p}}_{n+1}) = (\mathbf{p}_1, \ldots, \mathbf{p}_n, \mathbf{p}_{n+1} + \mathbf{l})$. By Proposition 2.2 we have

$$(2.11)\quad \mathbf{D}_{\overline{\nabla}}\theta(\gamma \underset{n+1}{\cdot} a = \Sigma_{i=1}^{n+1}(-1)^{i+1+\underline{\xi}_i}\underline{F}^i\overleftarrow{\mathbf{D}}_{\underline{\mathbf{p}}_i})(0)$$

$$(2.12)\quad \mathbf{D}_{\overline{\nabla}}\theta(\gamma) = \Sigma_{i=1}^{n+1}(-1)^{i+1+\xi_i}(F^i\overleftarrow{\mathbf{D}}_{\mathbf{p}_i})(0)$$

where $\underline{\xi}_i = \underline{\mathbf{p}}_i\Sigma_{h>i}\underline{\mathbf{p}}_h$ and $\underline{\mathbf{F}}^i(e) = q^{\overline{\nabla}}_{(\zeta\circ\gamma_i,\mathrm{e})}(\theta(\underline{\gamma}^i(e)))$ for any $e \in \mathbf{D}^{\underline{\mathbf{p}}_i}$ with $\underline{\gamma}^i(e)(d_1, \ldots, d_n) = \gamma(d_1, \ldots, d_{i-1}, e, d_i, \ldots, d_n a)$ for any $(d_1, \ldots, d_n) \in D^{\underline{\mathbf{p}}_1} \times \ldots \times D^{\underline{\mathbf{p}}_{i-1}} \times D^{\mathbf{p}_{i+1}} \times \ldots \times D^{\underline{\mathbf{p}}_{n+1}}$ $(1 \le i \le n)$ and $\underline{\gamma}^{n+1}(e)(d_1, \ldots, d_n) = \gamma(d_1, \ldots, d_n, ae)$ for any $(d_1, \ldots, d_n) \in D^{\mathbf{p}_i} \times \ldots \times D^{\mathbf{p}_n}$ while $\xi_i = \mathbf{p}_i\Sigma_{h>i}\mathbf{p}_h$ and $F^i(e) = q^{\overline{\nabla}}_{(\zeta\circ\gamma_i,e)}(\theta(\gamma^i(e)))$ for any $e \in D^{\mathbf{p}_i}$ with $\gamma^i(e)(d_1, \ldots, d_n) = \gamma(d_1, \ldots, d_{i-1}, \mathrm{e}, d_i, \ldots, d_n)$ for any $(d_1, \ldots, d_n) \in D^{\mathbf{p}_1} \times \ldots \times D^{\mathbf{p}_{i-1}} \times D^{\mathbf{p}_{i+1}} \times \ldots \times D^{\mathbf{p}_{n+1}}$ $(1 \le i \le n + 1)$. For any natural number i with $1 \le i \le n$ and any $e \in D^{\mathbf{p}_i}$ we have

$$\begin{aligned}(2.13)\quad \underline{F}^i\overleftarrow{\mathbf{D}}_{\mathbf{p}_i}(0)e &= \underline{F}^i(e) - \underline{F}^i(0)\\ &= (F^i(e) - F^i(0))a\\ &= F^i\overleftarrow{\mathbf{D}}_{\mathbf{p}_i}(0)ea\\ &= (-1)^{\mathbf{p}_i}F^i\overleftarrow{\mathbf{D}}_{\mathbf{p}_i}(0)ae\end{aligned}$$

so that

$$(2.14)\quad \underline{F}^i\overleftarrow{\mathbf{D}}_{\mathbf{p}_i}(0) = (-1)^{\mathbf{p}_i}F^i\overleftarrow{\mathbf{D}}_{\mathbf{p}_i}(0)a$$

On the other hand, for any $e \in D^{\underline{\mathbf{p}}_{n+1}}$, we have

$$(2.15)\quad \underline{F}^{n+1}\overleftarrow{\mathbf{D}}_{\underline{\mathbf{p}}_{n+1}}(0) = F^{n+1}\overleftarrow{\mathbf{D}}_{\mathbf{p}_{n+1}}(0)a$$

Since $\underline{\xi}_i = \xi_i + \mathbf{p}_i$ $(1 \le i \le n)$ and $\underline{\xi}_{n+1} = \xi_{n+1} = 0$, the desired equality (2.10) follows from (2.11), (2.12), (2.14), and (2.15). ■

Lemma 2.4. We have

$$(2.16)\quad \mathbf{D}_{\overline{\nabla}}\theta(\gamma \underset{i}{\cdot} a) = \mathbf{D}_{\overline{\nabla}}\theta(a \underset{i+1}{\cdot} \gamma) \qquad (1 \le i \le n)$$

for any $(\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}) \in (\mathbb{Z}_2)^{n+1}$, any $\gamma \in \mathbf{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_{n+1}}M$, and any $a \in \mathbb{R}_{\mathbf{o}}$.

Proof. Let $(\underline{\mathbf{p}}_1, \ldots, \underline{\mathbf{p}}_i \ldots, \underline{\mathbf{p}}_{n+1}) = (\mathbf{p}_1, \ldots, \mathbf{p}_i + \mathbf{l}, \ldots, \mathbf{p}_{n+1})$, and $(\overline{\mathbf{p}}_1, \ldots, \overline{\mathbf{p}}_{i+1}, \ldots, \overline{\mathbf{p}}_{n+1}) = (\mathbf{p}_1, \ldots, \mathbf{p}_{i+1} + \mathbf{l}, \ldots, \mathbf{p}_{n+1})$. Let ξ_j and F^j

$(1 \le j \le n+1)$ be the same as in the previous lemma. By Proposition 2.2 we have

$$\text{(2.17)}\quad \mathbf{D}_{\triangledown}\theta(\gamma \underset{i}{;} a) = \Sigma_{j=1}^{n+1}(-1)^{j+1+\underline{\xi}_j}(\underline{F}^j \overleftarrow{\mathbf{D}}_{\underline{\mathbf{p}}_j})(0)$$

$$\text{(2.18)}\quad \mathbf{D}_{\triangledown}\theta(a \underset{i+1}{;} \gamma) = \Sigma_{j=1}^{n+1}(-1)^{j+1+\overline{\xi}_j}(\overline{F}^j \overleftarrow{\mathbf{D}}_{\overline{\mathbf{p}}_j})(0)$$

where $\underline{\xi}_j = \underline{\mathbf{p}}_j \Sigma_{h>j} \underline{\mathbf{p}}_h$ and $\underline{F}^j(e) = q^{\overline{\triangledown}}_{(\xi\circ\gamma_j,e)}(\theta(\underline{\gamma}^j(e)))$ for any $e \in D_{\underline{\mathbf{p}}_j}$ with $\underline{\gamma}^j(e)(d_1, \ldots, d_n) = (\gamma_i a)(d_1, \ldots, d_{j-1}, e, d_j, \ldots, d_n)$ for any $(d_1, \ldots, d_n) \in D^{\underline{\mathbf{p}}_1} \times \ldots \times D^{\underline{\mathbf{p}}_{j-1}} \times D^{\underline{\mathbf{p}}_{j+1}} \times \ldots \times D^{\underline{\mathbf{p}}_{n+1}}$ $(1 \le j \le n+1)$, while $\overline{\xi}^j = \overline{\mathbf{p}}_j \Sigma_{h>j} \overline{\mathbf{p}}_h$ and $\overline{F}^j(e) = q^{\overline{\triangledown}}_{(\zeta\circ\gamma_j,e)}(\theta(\overline{\gamma}^j(e)))$ for any $e \in D^{\overline{\mathbf{p}}}j$ with $\overline{\gamma}^j(e)\ (d_1, \ldots, d_n) = (a \underset{i+1}{;} \gamma)\ (d_1, \ldots, d_{j-1}, d, d_j, \ldots, d_n)$ for any $(d_1, \ldots, d_n) \in D^{\overline{\mathbf{p}}_1} \times \ldots \times D^{\overline{\mathbf{p}}_{j-1}} \times D^{\overline{\mathbf{p}}_{j+1}} \times \ldots \times D^{\overline{\mathbf{p}}_{n+1}}$ $(1 \le j \le n+1)$. For any j with $j \ne i$ and $j \ne i+1$,

$$\text{(2.19)}\quad \begin{aligned} \underline{F}^j \overleftarrow{\mathbf{D}}_{\mathbf{p}_j}(0)e &= F^j(e) - F^j(0) \\ &= \overline{F}^j(e) - \overline{F}^j(0) \\ &= \overline{F}^j \overleftarrow{\mathbf{D}}_{\mathbf{p}_j}(0)e \end{aligned}$$

so that

$$\text{(2.20)}\quad \underline{F}^j \overleftarrow{\mathbf{D}}_{\mathbf{p}_j}(0) = \overline{F}^j \overleftarrow{\mathbf{D}}_{\mathbf{p}_j}(0)$$

For $j = i$ we have that for any $e \in D^{\mathbf{p}_i}$.

$$\text{(2.21)}\quad \begin{aligned} \underline{F}^j \overleftarrow{\mathbf{D}}_{\underline{\mathbf{p}}_i}(0)e &= F^j \overleftarrow{\mathbf{D}}_{\mathbf{p}_i}(0)ae \\ &= (-1)^{\mathbf{p}_i} F^j \overleftarrow{\mathbf{D}}_{\mathbf{p}_i}(0)ea \\ &= (-1)^{\mathbf{p}_i}\{F^i(e) - F^i(0)\}a \\ &= (-1)^{\Sigma_{h\ge i}\mathbf{p}_h}\{\overline{F}^i(e) - \overline{F}^i(0)\} \\ &= (-1)^{\Sigma_{h\ge i}\mathbf{p}_h} \overline{F}^i \overleftarrow{\mathbf{D}}_{\overline{\mathbf{p}}_i}(0)e \end{aligned}$$

so that

$$\text{(2.22)}\quad \underline{F}^j \overleftarrow{\mathbf{D}}_{\underline{\mathbf{p}}_i}(0) = (-1)^{\Sigma_{h\ge i}\mathbf{p}_h} \overline{F}^i \overleftarrow{\mathbf{D}}_{\overline{\mathbf{p}}_i}(0)$$

For $j = i+1$ we have that for any $e \in D^{\overline{\mathbf{p}}_{i+1}}$,

$$
\begin{aligned}
(2.23)\quad \overline{F}^{i+1}\overleftarrow{\mathbf{D}}_{\overline{\mathbf{p}}_{i+1}}(0)e &= (-1)^{\overline{\mathbf{p}}_{i+1}} F^{i+1}\overleftarrow{\mathbf{D}}_{\mathbf{p}_{i+1}}(0)ae \\
&= F^{i+1}\overleftarrow{\mathbf{D}}_{\mathbf{p}_{i+1}}(0)ea \\
&= \{F^{i+1}(e) - F^{i+1}(0)\}a \\
&= (-1)^{\Sigma_{h>i+1}\mathbf{p}_h}\{\underline{F}^{i+1}(e) - \underline{F}^{i+1}(0)\} \\
&= (-1)^{\Sigma_{h>i+1}\mathbf{p}_h}\underline{F}^{i}\overleftarrow{\mathbf{D}}_{\overline{\mathbf{p}}_i}(0)e
\end{aligned}
$$

so that

$$(2.24)\quad \overline{F}^{i+1}\overleftarrow{\mathbf{D}}_{\overline{\mathbf{p}}_{i+1}}(0) = (-1)^{\Sigma_{h>i+1}\mathbf{p}_h} F^{i}\overleftarrow{\mathbf{D}}_{\overline{\mathbf{p}}_i}(0)$$

For j with $j < i$ we have

$$(2.25)\quad \underline{\xi}_j = \overline{\xi}_j = \mathbf{p}_j + \xi_j$$

while for j with $j > \mathrm{i} + 1$ we have

$$(2.26)\quad \underline{\xi}_j = \overline{\xi}_j = \xi_j$$

On the other hand, we have

$$
\begin{aligned}
(2.27)\quad \underline{\xi}_i &= \xi_i + \textstyle\sum_{h>i} \mathbf{p}_h \\
&= \overline{\xi}_i + \textstyle\sum_{h\geq i} \mathbf{P}_h \\
(2.88)\quad \overline{\xi}_{i+1} &= \xi_{i+1} + \textstyle\sum_{h>i+1} \mathbf{p}_h \\
&= \underline{\xi}_{i+1} + \textstyle\sum_{h>i+1} \mathbf{p}_h
\end{aligned}
$$

Therefore our desired (2.16) follows from (2.17), (2.18), (2.20), (2.22), (2.24), and (2.25)–(2.28). ■

3. INDUCED SUPERCONNECTIONS

Now we define some induced superconnections. Let ζ: $E \to M$ and η: $F \to M$ be supervector bundles over the same base space M with superconnections ∇ and ∇' bestowed upon them. First we define an induced superconnection $\nabla \oplus \nabla'$ on the Whitney sum $\zeta \oplus \eta$ as follows:

(3.1) $(\nabla \oplus \nabla')(\mathbb{t}, v_\zeta \oplus v_\eta)(\mathbb{d})$
$= \nabla(\mathbb{t}, \mathrm{v}_\zeta)(\mathbb{d}) \oplus \nabla'(\mathbb{t}, \mathrm{v}_\eta)(\mathbb{d})$
for any $\mathbb{t} \in M^{D(\mathbf{0},\mathbf{1})}$, any $v_\zeta \in E_{\mathbb{t}(0)}$, any $v_\eta \in F_{\mathbb{t}(0)}$,
and any $\mathbb{d} \in D(\mathbf{0}, \mathbf{1})$.

Proposition 3.1. For any $\bar{\mathbb{t}}_\zeta \in E^{D(\mathbf{0},\mathbf{1})}$ and any $\bar{\mathbb{t}}_\eta \in F^{D(\mathbf{0},\mathbf{1})}$ with $\zeta^{D(\mathbf{0},\mathbf{1})}(\bar{\mathbb{t}}_\zeta) = \eta^{D(\mathbf{0},\mathbf{1})}(\bar{\mathbb{t}}_\eta)$, we have

$$(3.2)\quad \omega_{\zeta\oplus\eta}(\bar{\mathbb{t}}_\eta \oplus \bar{\mathbb{t}}_\zeta) = \omega_\zeta(\bar{\mathbb{t}}_\zeta) \oplus \omega_\eta(\bar{\mathbb{t}}_\eta)$$

where $\omega_{\zeta\oplus\eta}$, ω_ζ and ω_η denote the superconnection forms of $\nabla \oplus \nabla'$, ∇, and ∇', respectively.

Proof. Let $\mathbb{t} = \zeta^{D(\mathbf{0},\mathbf{1})}(\bar{\mathbb{t}}_\zeta = \eta^{D(\mathbf{0},\mathbf{1})}(\bar{\mathbb{t}}_\eta)$. For any $\mathbb{d} \in D(\mathbf{0}, \mathbf{1})$, we have, by Proposition 2.2, that

$$\begin{aligned}(3.3)\quad & q^{\nabla\oplus\nabla'}_{(\mathbb{t},\mathbb{d})}(\bar{\mathbb{t}}_\zeta(\mathbb{d}) \oplus \bar{\mathbb{t}}_\eta(\mathbb{d}))\\ &= (\bar{\mathbb{t}}_\zeta(0) + \omega_{\zeta,\mathbf{e}}(\bar{\mathbb{t}}_\zeta)\mathbb{d}_\mathbf{e} + \omega_{\zeta,\mathbf{o}}(\bar{\mathbb{t}}_\zeta)\mathbb{d}_\mathbf{o})\\ &\quad \oplus (\bar{\mathbb{t}}_\eta(0) + \omega_{\eta,\mathbf{e}}(\bar{\mathbb{t}}_\zeta)\mathbb{d}_\mathbf{e} + \omega_{\eta,\mathbf{o}}(\bar{\mathbb{t}}_\zeta)\mathbb{d}_\mathbf{o})\\ &= (\bar{\mathbb{t}}_\zeta(0) \oplus (\bar{\mathbb{t}}_\eta(0)) + (\omega_{\zeta,\mathbf{e}}(\bar{\mathbb{t}}_\zeta) \oplus \omega_{\eta,\mathbf{e}}(\bar{\mathbb{t}}_\eta))\mathbb{d}_\mathbf{e}\\ &\quad + (\omega_{\zeta,\mathbf{o}}(\bar{\mathbb{t}}_\zeta) \oplus \omega_{\eta,\mathbf{o}}(\bar{\mathbb{t}}_\eta))\mathbb{d}_\mathbf{o}\end{aligned}$$

Therefore the desired proposition obtains by Proposition 2.2 again. ■

Corollary 3.2. For any $\mu \in \operatorname{Sec}\zeta$ and any $\nu \in \operatorname{Sec}\eta$, we have

$$(3.4)\quad \mathbf{D}_{\nabla\oplus\nabla'}(\mu \oplus \nu) = \mathbf{D}_\nabla\mu \oplus \mathbf{D}_{\nabla'}\nu$$

We now define an induced superconnection $\hat{\nabla}$ on $\pi_{\mathscr{L}(\zeta,\eta)}$ as follows:

$$(3.5)\quad \hat{\nabla}(\mathbb{t}, \hat{v})(\mathbb{d})(v) = p^{\nabla'}_{(\mathbb{t},\mathbb{d})}(\hat{v}(q^{\nabla}_{(\mathbb{t},\mathbb{d})}(v)))$$
for any $\mathbb{t} \in \mathrm{M}^{D(\mathbf{0},\mathbf{1})}$, any $\mathbb{d} \in D(\mathbf{0}, \mathbf{1})$, any $\hat{v} \in \mathscr{L}(\zeta, \eta)_{\mathbb{t}(0)}$, and any $v \in \mathrm{F}_{\mathbb{t}(0)}$

Proposition 3.3. For any $\hat{\mathbb{t}} \in \mathscr{L}(\zeta, \eta)^{D(\mathbf{0},\mathbf{1})}$, and any $\bar{\mathbb{t}} \in E^{D(\mathbf{0},\mathbf{1})}$ with $(\pi_{\mathscr{L}(\zeta,\eta)})^{D(\mathbf{0},\mathbf{1})}(\hat{\mathbb{t}}) = \zeta^{D(\mathbf{0},\mathbf{1})}(\bar{\mathbb{t}})$, we have

$$\begin{aligned}(3.6)\quad \omega_\eta(\hat{\mathbb{t}}(\bar{\mathbb{t}})) &= \hat{\omega}_\mathbf{e}(\hat{\mathbb{t}})(\bar{\mathbb{t}}(0)) + \hat{\omega}_\mathbf{o}(\hat{\mathbb{t}})(\bar{\mathbb{t}}(0)_\mathbf{e})\\ &\quad - \hat{\omega}_\mathbf{o}(\hat{\mathbb{t}})(\bar{\mathbb{t}}(0)_\mathbf{o}) + \hat{\mathbb{t}}(0)(\omega_\zeta(\bar{\mathbb{t}}))\end{aligned}$$

where $\hat{\omega}$ denotes the superconnection form of $\hat{\nabla}$ and $\hat{\mathbb{t}}(\bar{\mathbb{t}})$ denotes the mapping $\mathbb{d} \in D(\mathbf{0}, \mathbf{1} \mapsto \hat{\mathbb{t}}(\mathbb{d})(\bar{\mathbb{t}}(\mathbb{d}))$.

Proof. Let $\mathbb{t} = (\pi_{\mathscr{L}(\zeta,\eta)})^{\mathrm{D}(\mathbf{0},\mathbf{1})}(\hat{\mathbb{t}}) = \zeta^{D(\mathbf{0},\mathbf{1})}(\bar{\mathbb{t}})$. For any $\mathrm{d} \in D(\mathbf{0}, \mathbf{1})$, we have, by Proposition 2.2, that

$$\begin{aligned}(3.7)\quad & q^{\nabla'}_{(\mathbb{t},\mathbb{d})}(\hat{\mathbb{t}}(\mathbb{d})(\bar{\mathbb{t}}(\mathbb{d})))\\ &= q^{\hat{\nabla}}_{(\mathbb{t},\mathbb{d})}(\hat{\mathbb{t}}(\mathbb{d}))(q^{\nabla}_{(\mathbb{t},\mathbb{d})}(\bar{\mathbb{t}}(\mathbb{d})))\\ &= (\hat{\mathbb{t}})(0) + \hat{\omega}_\mathbf{e}(\hat{\mathbb{t}})\mathbb{d}_\mathbf{e} + \hat{\omega}_\mathbf{o}(\hat{\mathbb{t}})\mathbb{d}_\mathbf{o})(\bar{\mathbb{t}}(0) + \omega_{\zeta,\mathbf{e}}(\bar{\mathbb{t}})\mathbb{d}_\mathbf{e}\end{aligned}$$

$$
\begin{aligned}
&\quad + \omega_{\zeta,\mathbf{o}}(\bar{\mathbb{t}})\mathbb{d}_{\mathbf{o}}) \\
&= (\hat{\mathbb{t}}(0) + \mathbb{d}_{\mathbf{e}}\hat{\omega}_{\mathbf{e}}(\hat{\mathbb{t}}) + \mathbb{d}_{\mathbf{o}}(\hat{\omega}_{\mathbf{o}}(\hat{\mathbb{t}}))_{\mathbf{e}} - \mathbb{d}_{\mathbf{o}}(\hat{\omega}_{\mathbf{o}}(\hat{\mathbb{t}}))_{\mathbf{o}})(\bar{\mathbb{t}}(0) \\
&\quad + \omega_{\zeta,\mathrm{e}}(\bar{\mathbb{t}})\mathbb{d}_{\mathbf{e}} + \omega_{\zeta,\mathrm{o}}(\bar{\mathbb{t}})\mathbb{d}_{\mathbf{o}}) \\
&= (\hat{\mathbb{t}}(0)(\bar{\mathbb{t}}(0)) + \mathbb{d}_{\mathbf{e}}\hat{\omega}_{\mathbf{e}}(\hat{\mathbb{t}})(\bar{\mathbb{t}}(0)) + \mathbb{d}_{\mathbf{o}}(\hat{\omega}_{\mathbf{o}}(\hat{\mathbb{t}}))_{\mathbf{e}}(\bar{\mathbb{t}}(0)) \\
&\quad - \mathbb{d}_{\mathbf{o}}(\hat{\omega}_{\mathbf{o}}(\hat{\mathbb{t}}))_{\mathbf{o}}(\bar{\mathbb{t}}(0)) + \hat{\mathbb{t}}(0)(\omega_{\zeta,\mathbf{e}}(\bar{\mathbb{t}}))\mathbb{d}_{\mathbf{e}} \\
&\quad + \hat{\mathbb{t}}(0)(\omega_{\zeta,\mathbf{o}}(\bar{\mathbb{t}}))\mathbb{d}_{\mathbf{o}} \\
&= \hat{\mathbb{t}}(0)(\bar{\mathbb{t}}(0)) + \hat{\omega}_{\mathbf{e}}(\hat{\mathbb{t}})(\bar{\mathbb{t}}(0))\mathbb{d}_{\mathbf{e}} + (\hat{\omega}_{\mathbf{o}}(\hat{\mathbb{t}}))_{\mathbf{e}}(\bar{\mathbb{t}}(0)_{\mathbf{e}})\mathbb{d}_{\mathbf{o}} \\
&\quad - \hat{\omega}_{\mathbf{o}}(\hat{\mathbb{t}}))_{\mathbf{e}}(\bar{\mathbb{t}}(0)_{\mathbf{o}})\mathbb{d}_{\mathbf{o}} + (\hat{\omega}_{\mathbf{o}}(\hat{\mathbb{t}}))_{\mathbf{o}}(\bar{\mathbb{t}}(0)_{\mathbf{e}})\mathbb{d}_{\mathbf{o}} \\
&\quad - (\hat{\omega}_{\mathbf{o}}(\hat{\mathbb{t}}))_{\mathbf{o}}(\bar{\mathbb{t}}(0)_{\mathbf{o}})\mathbb{d}_{\mathbf{o}} + \hat{\mathbb{t}}(0)(\omega_{\zeta,\mathbf{e}}(\bar{\mathbb{t}}))\mathbb{d}_{\mathbf{e}} \\
&\quad + \hat{\mathbb{t}}(0)(\omega_{\zeta,\mathbf{o}}(\bar{\mathbb{t}}))\mathbb{d}_{\mathbf{o}} \\
&= \hat{\mathbb{t}}(0)(\bar{\mathbb{t}}(0)) + \{\hat{\omega}_{\mathbf{e}}(\hat{\mathbb{t}})(\bar{\mathbb{t}}(0)) + \hat{\mathbb{t}}(0)(\omega_{\zeta,\mathbf{e}}(\bar{\mathbb{t}}))\}\mathbb{d}_{\mathbf{e}} \\
&\quad + \{\hat{\omega}_{\mathbf{o}}(\hat{\mathbb{t}}))_{\mathbf{e}}(\bar{\mathbb{t}}(0)_{\mathbf{e}}) - \hat{\omega}_{\mathrm{o}}(\hat{\mathbb{t}}))_{\mathbf{e}}(\bar{\mathbb{t}}(0)_{\mathbf{o}}) \\
&\quad + \hat{\omega}_{\mathbf{o}}(\hat{\mathbb{t}}))_{\mathbf{o}}(\bar{\mathbb{t}}(0)_{\mathbf{e}}) - \hat{\omega}_{\mathrm{o}}(\hat{\mathbb{t}}))_{\mathbf{o}}(\bar{\mathbb{t}}(0)_{\mathbf{o}}) \\
&\quad + \hat{\mathbb{t}}(0)(\omega_{\zeta,\mathbf{o}}(\bar{\mathbb{t}}))\}\mathbb{d}_{\mathbf{o}} \\
&= \hat{\mathbb{t}}(0)(\bar{\mathbb{t}}(0)) + \{\hat{\omega}_{\mathbf{e}}(\hat{\mathbb{t}})(\bar{\mathbb{t}}(0)) + \hat{\mathbb{t}}(0)(\omega_{\zeta,\mathbf{e}}(\bar{\mathbb{t}}))\}\mathbb{d}_{\mathbf{e}} \\
&\quad + \{\hat{\omega}_{\mathbf{o}}(\hat{\mathbb{t}})(\bar{\mathbb{t}}(0)_{\mathbf{e}}) - \hat{\omega}_{\mathbf{o}}(\hat{\mathbb{t}})(\bar{\mathbb{t}}(0)_{\mathbf{o}}) + \hat{\mathbb{t}}(0)(\omega_{\zeta,\mathbf{o}}(\bar{\mathbb{t}}))\}\mathbb{d}_{\mathbf{o}}
\end{aligned}
$$

Therefore the desired proposition obtains by Proposition 2.2 again. ∎

Corollary 3.4. For any $\mu \in \mathrm{Sec}\ \zeta$ and any $\iota \in \mathrm{Sec}\ \pi_{\mathcal{L}(\zeta,\eta)}$, we have

$$
\begin{aligned}
\text{(3.8)} \quad \mathbf{D}_{\nabla'}(\iota(\mu)) &= (\mathbf{D}^{\mathbf{e}}_{\hat{\nabla}}\iota)(\mu) + (\mathbf{D}^{\mathbf{o}}_{\hat{\nabla}}\iota)(\mu_{\mathbf{e}}) - (\mathbf{D}^{\mathbf{o}}_{\hat{\nabla}}\iota)(\mu_{\mathbf{o}}) \\
&\quad + \iota(\mathbf{D}_{\nabla}\mu)
\end{aligned}
$$

If η is the trivial bundle $M \times \mathbb{R} \to M$ and the superconnection ∇' is trivial, then the superconnection $\hat{\nabla}$ is usually denoted by ∇^*. If $\zeta = \eta$ and $\nabla = \nabla'$, then the superconnection $\hat{\nabla}$ is usually denoted by $\tilde{\nabla}$.

4. CURVATURE

Let $\zeta: E \to M$ be a supervector bundle endowed with a superconnection ∇, which shall be fixed throughout this section. The principal objective of this section is to introduce a sort of curvature abiding by the so-called second Bianchi identity. First let us introduce a preliminary version of curvature somewhat disobedient to the second Bianchi identity, from which our desired curvature naturally follows. The connection form ω is surely an element of $\Xi_1(E \xrightarrow{\zeta} M; \zeta)$, and its covariant exterior derivative $\mathbf{D}_\nabla\omega \in \Xi_2(E \xrightarrow{\zeta} M; \zeta)$ is called the *curvature form of the first kind* and denoted by Ω, for which we have the following result.

Proposition 4.1. For any $\overline{\gamma} \in E^{D(\mathbf{p})\times D(\mathbf{q})}$ and any $(d_1, d_2) \in \mathrm{D}(\mathbf{p}) \times \mathrm{D}(\mathbf{q})$ with $\gamma = \zeta \circ \overline{\gamma}$, $t_1 = \gamma(\cdot, 0)$, $t_2 = \gamma(d_1, \cdot)$, $t_3 = \gamma\,(0, \cdot)$, and $t_4 = \gamma\,(\cdot, d_2)$, we have

$$\begin{aligned}(4.1)\quad & (-1)^{\mathbf{pq}}\Omega(\overline{\gamma})d_1d_2 \\ &= q_{(t_1,d_1)} \cdot q_{(t_2,d_2)}(\overline{\gamma}(d_1, d_2)) \\ &\quad - q_{(t_3,d_2)} \cdot q_{(t_4,d_1)}(\overline{\gamma}(d_1, d_2))\end{aligned}$$

Proof. By the very definition of covariant exterior differentiation, we have

$$\begin{aligned}(4.2)\quad & (-1)^{\mathbf{pq}}\Omega(\overline{\gamma})d_1d_2 \\ &= \omega(\overline{\gamma}(\cdot, 0))d_1 + q_{(t_1,d_1)}(\omega(\overline{\gamma}(d_1, \cdot)))d_2 \\ &\quad - q_{(t_3,d_2)}(\omega(\overline{\gamma}(\cdot, d_2)))d_1 - \omega(\overline{\gamma}(0, \cdot))d_2\end{aligned}$$

By Proposition 2.2 we have

$$\begin{aligned}(4.3)\quad & \omega(\overline{\gamma}(\cdot, 0))d_1 \\ &= q_{(t_1,d_1)}(\overline{\gamma}(d_1, 0)) - \overline{\gamma}(0, 0)\end{aligned}$$

$$\begin{aligned}(4.4)\quad & q_{(t_1,d_1)}(\omega(\overline{\gamma}(d_1, \cdot)))d_2 \\ &= q_{(t_1,d_1)}\{q_{(t_2,d_2)}(\overline{\gamma}(d_1, d_2)) - \overline{\gamma}(d_1, 0)\} \\ &= q_{(t_1,d_1)} \circ q_{(t_2,d_2)}(\overline{\gamma}(d_1, d_2)) - q_{(t_1,d_1)}(\overline{\gamma}(d_1, 0))\end{aligned}$$

$$\begin{aligned}(4.5)\quad & q_{(t_3,d_2)}(\omega(\overline{\gamma}(\cdot, d_2)))d_1 \\ &= q_{(t_3,d_2)}\{q_{(t_4,d_1)}(\overline{\gamma}(d_1, d_2)) - \overline{\gamma}(0, d_2)\} \\ &= q_{(t_3,d_2)} \circ q_{(t_4,d_1)}(\overline{\gamma}(d_1, d_2)) - q_{(t_3,d_2)}(\overline{\gamma}(0, d_2))\end{aligned}$$

$$\begin{aligned}(4.6)\quad & \omega(\overline{\gamma}(0, \cdot))d_2 \\ &= q_{(t_3,d_2)}(\overline{\gamma}(0, d_2)) - \overline{\gamma}(0, 0)\end{aligned}$$

Therefore the desired conclusion follows. ■

Now we introduce another curvature form, to be called the *curvature form of the second kind* and to be denoted by $\tilde{\Omega}$, as follows:

(4.7) $\tilde{\Omega}(\overline{\gamma}) = \Omega(h(\overline{\gamma}))$ for any microsquare $\overline{\gamma}$ on E.

where $h(\overline{\gamma})$ denotes the *horizontal, component* of $\overline{\gamma}$ (Moerdijk and Reyes, 1991, Chapter V, §6) in the sense that for any $(\mathbb{d}_1, \mathbb{d}_2) \in D(\mathbf{0}, \mathbf{1})^2$,

(4.8) $h(\overline{\gamma})(\mathbb{d}_1, \mathbb{d}_2) = p_{(\gamma(\mathbb{d}_1,\cdot),\mathbb{d}_2)} \circ p_{(\gamma(\cdot,0),\mathbb{d}_1)}(\overline{\gamma}(0, 0))$

with $\gamma = \zeta \circ \overline{\gamma}$. For the curvature form of the second kind, we have the following result.

Proposition 4.2. Using the same notation as in Proposition 4.1, we have

(4.9)
$$
\begin{aligned}
&(-1)^{\mathbf{pq}}\tilde{\Omega}(\overline{\gamma})d_1d_2 \\
&= \overline{\gamma}(0, 0) \\
&\quad - q_{(t_3,d_2)} \circ q_{(t_4,d_1)} \circ p_{(t_2,d_2)} \circ p_{(t_1,d_1)}(\overline{\gamma}(0, 0))
\end{aligned}
$$

so that $\tilde{\Omega}(\overline{\gamma})$ depends only on $\gamma = \zeta\cdot\overline{\gamma}$ and $v = \overline{\gamma}(0, 0)$, which enables us to regard $\tilde{\Omega}$ as a function from $\mathbf{T}^2(M)$ to $\mathscr{L}(\zeta)$ in the sense that $\tilde{\Omega}(\gamma)(v) = \tilde{\Omega}(\overline{\gamma})$.

Proof. Simply put $h(\overline{\gamma})$ in place of $\overline{\gamma}$ in Proposition 4.1. ■

We now reckon $\tilde{\Omega}$ as a function from $M^{D(\mathbf{0},\mathbf{1})^2}$ to $\mathscr{L}(\zeta)$ in the canonical way, for which we have the following result.

Proposition 4.3. The function $\tilde{\Omega}$: $M^{D(\mathbf{0},\mathbf{1})^2} \to \mathscr{L}(\zeta)$ is a differential 2-form on M with values in $\pi_{\mathscr{L}(\xi)}$, i.e., $\tilde{\Omega} \in \Xi_2(M; \pi_{\mathscr{L}(\zeta)})$.

Proof. We define a function $\mathfrak{h}$: $M^{D(\mathbf{0},\mathbf{1})^2} \underset{M}{\times} E \to E^{D(\mathbf{0},\mathbf{1})^2}$ as follows:

(4.10) $\mathfrak{h}(\gamma, v)(\mathbb{d}_1, \mathbb{d}_2) = p_{(\gamma(\mathbb{d}_1,\cdot),\mathbb{d}_2)} \circ p_{(\gamma(\cdot,0),\mathbb{d}_1)}(v)$

for any $(\gamma, v) \in M^{D(\mathbf{0},\mathbf{1})^2} \underset{M}{\times} E$ and any $(\mathbb{d}_1, \mathbb{d}_2) \in D(\mathbf{0}, \mathbf{1})^2$

Then it is easy to see that

(4.11) $\mathfrak{h}(\gamma \underset{i}{\cdot} a, v) = \mathfrak{h}(\gamma, v) \underset{i}{\cdot} a$ for any $a \in \mathbb{R}$ $(i = 1, 2)$

Since $\tilde{\Omega}(\gamma)(v) = \Omega(\mathfrak{h}(\gamma, v))$ and Ω is 2-homogeneous, $\tilde{\Omega}$ is also 2-homogeneous. Now we use the same notation as in Propositions 4.1 and 4.2. To show that $\tilde{\Omega}$ is super alternating, we let $v_0 = v$ and define v_1 and v_2 in order as follows:

$$(4.12)\quad v_1 = q_{(t_3,d_2)} \cdot q_{(t_4,d_1)} \cdot p_{(t_2,d_2)} \cdot p_{(t_1,d_1)}(v_0)$$

$$(4.13)\quad v_2 = q_{(t_1,d_1)} \cdot q_{(t_2,d_2)} \cdot p_{(t_4,d_1)} \cdot p_{(t_3,d_2)}(v_1)$$

On the one hand, it follows directly from (4.12) and (4.13) that

$$(4.14)\quad v_2 = v_0$$

On the other hand, we can calculate v_1 and v_2 in order by making use of Proposition 4.2:

$$\begin{aligned}
(4.15)\quad v_1 &= v_0 - (-1)^{\mathbf{pq}}\tilde{\Omega}(\gamma)(v_0)d_1d_2 \\
(4.16)\quad v_2 &= v_1 - (-1)^{\mathbf{pq}}\tilde{\Omega}(\Sigma(\gamma))(v_1)d_2d_1 \\
&= v_0 - (-1)^{\mathbf{pq}}\tilde{\Omega}(\gamma)(v_0)d_1d_2 \\
&\quad - (-1)^{\mathbf{pq}}\tilde{\Omega}(\Sigma(\gamma))(v_0 - (-1)^{\mathbf{pq}}\tilde{\Omega}(\gamma)(v_0)d_1d_2)d_2d_1 \\
&\quad [(4.15)] \\
&= v_0 - (-1)^{\mathbf{pq}}\tilde{\Omega}(\gamma)(v_0)d_1d_2 \\
&\quad - (-1)^{\mathbf{pq}}\tilde{\Omega}(\Sigma(\gamma))(v_0)d_2d_1
\end{aligned}$$

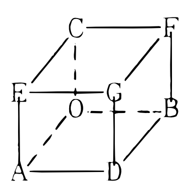
It follows from (4.14) and (4.16) that

$$(4.17)\quad \tilde{\Omega}(\gamma)(v_0) + (-1)^{\mathbf{pq}}\tilde{\Omega}(\Sigma(\gamma))(v_0) = 0$$

which means that $\tilde{\Omega}$ is super alternating. ■

Now we give a super, cubical version of Kock's (1996, Theorem 2) simplicial and combinatorial Bianchi identity.

Theorem 4.4. Let $\gamma \in M^{D(\mathbf{p})\times D(\mathbf{q})\times D(\mathbf{r})}$. Let $(d_1, d_2, d_3) \in D(\mathbf{p}) \times D(\mathbf{q}) \times D(\mathbf{r})$. We denote points $\gamma(0, 0, 0)$, $\gamma(d_1, 0, 0)$, $\gamma(0, d_2, 0)$, $\gamma(0, 0, d_3)$, $\gamma(d_1, d_2, 0)$, $\gamma(d_1, 0, d_3)$, $\gamma(0, d_2, d_3)$, and $\gamma(d_1, d_2, d_3)$ by O, A, B, C, D, E, F, and G respectively. These eight points are depicted figuratively as the eight vertices of a cube:



Then we have

(4.18) $P_{\mathrm{AO}} \circ P_{\mathrm{DA}} \circ P_{\mathrm{GD}} \circ R_{\mathrm{GFBD}} \circ R_{\mathrm{GECF}} \circ R_{\mathrm{GDAE}} \circ P_{\mathrm{DG}} \circ P_{\mathrm{AD}} \circ P_{\mathrm{OA}} \circ R_{\mathrm{OCEA}} \circ R_{\mathrm{OBFC}} \circ R_{\mathrm{OADB}} = \mathrm{id}_{\mathrm{O}}$

where

(4.19) For any adjacent vertices X and Y of the cube, P_{XY} denotes the parallel transport from X to Y along the line connecting X and Y (e.g., P_{OA} and P_{AO} denote $p_{(\gamma(\cdot,0),d_1)}$ and $q_{(\gamma(\cdot,0),d_1)}$ respectively).

(4.20) For any four vertices X, Y, Z, and W of the cube rounding one of the six facial squares of the cube, R_{XYZW} denotes $P_{\mathrm{WX}} \circ P_{\mathrm{ZW}} \circ P_{\mathrm{YZ}} \circ P_{\mathrm{XY}}$ (e.g., R_{OADB} denotes $q_{(\gamma(0,\cdot,0),d_2)} \circ q_{(\gamma(\cdot,d_2,0),d_1)} \circ p_{(\gamma(d_1,\cdot,0),d_2)} \circ p_{(\gamma(\cdot,0,0),d_1)}$).

(4.21) id_{O} is the identity transformation of E_{O}.

Proof. Write (4.18) exclusively in terms of P_{XY}'s, and write off all consecutive $P_{\mathrm{XY}} \circ P_{\mathrm{YX}}$'s. ■

The above theorem gives rise to the following form of the second Bianchi identity in our super context.

Theorem 4.5. We have

(4.22) $\mathbf{D}_{\tilde{\nabla}}\, \tilde{\Omega} = 0$

where $\mathbf{D}_{\tilde{\nabla}}$ is the covariant exterior differentiation with respect to the induced superconnection $\tilde{\nabla}$ on $\pi_{\mathcal{L}(\xi)}$, and recall that $\tilde{\Omega} \in \Xi_2, (M; \pi_{\mathcal{L}(\xi)})$, as was explained in Proposition 4.3.

Proof. The proof is carried out by the same method as in Proposition 4.3. Let γ, d_1, d_2, d_3, O, A, B, C, D, E, F, and G be as in Theorem 4.4. Given $v_0 \in E_{\gamma(0,0,0)}$, we define $v_i \in E_{\gamma(0,0,0)}$ $(i = 1, 2, 3, 4, 5, 6)$ in order as follows:

(4.23) $v_1 = R_{\mathrm{OADB}}(v_0)$

(4.24) $v_2 = R_{\mathrm{OBFC}}(v_1)$

(4.25) $v_3 = R_{\mathrm{OCEA}}(v_2)$

(4.26)
$$\begin{aligned} v_4 &= P_{\mathrm{AO}} \circ P_{\mathrm{DA}} \circ P_{\mathrm{GD}} \circ R_{\mathrm{GDAE}} \circ P_{\mathrm{DG}} \circ P_{\mathrm{AD}} \circ P_{\mathrm{OA}}(v_3) \\ &= P_{\mathrm{AO}} \circ R_{\mathrm{AEGD}} \circ P_{\mathrm{OA}}(v_3) \end{aligned}$$

(4.27)
$$\begin{aligned} v_5 &= P_{\mathrm{AO}} \circ P_{\mathrm{DA}} \circ P_{\mathrm{GD}} \circ R_{\mathrm{GECF}} \circ P_{\mathrm{DG}} \circ P_{\mathrm{AD}} \circ P_{\mathrm{OA}}(v_4) \\ &= P_{\mathrm{AO}} \circ R_{\mathrm{AEGD}} \circ P_{\mathrm{EA}} \circ R_{\mathrm{ECFG}} \circ P_{\mathrm{AE}} \circ R_{\mathrm{ADGE}} \circ P_{\mathrm{OA}}(v_4) \\ &= P_{\mathrm{AO}} \circ P_{\mathrm{EA}} \circ R_{\mathrm{EGDA}} \circ R_{\mathrm{ECFG}} \circ R_{\mathrm{EADC}} \circ P_{\mathrm{AE}} \circ P_{\mathrm{OA}}(v_4) \end{aligned}$$

$$= R_{\mathrm{OCEA}} \circ P_{\mathrm{CO}} \circ P_{\mathrm{EC}} \circ R_{\mathrm{EGDA}} \circ R_{\mathrm{ECFG}} \circ R_{\mathrm{EADG}} \circ P_{\mathrm{CE}} \circ P_{\mathrm{OC}} \circ R_{\mathrm{OAEC}}(v_4)$$
$$= R_{\mathrm{OCEA}} \circ P_{\mathrm{CO}} \circ P_{\mathrm{EC}} \circ R_{\mathrm{EGDA}} \circ P_{\mathrm{CE}} \circ R_{\mathrm{CFGE}} \circ P_{\mathrm{EC}} \circ R_{\mathrm{EADG}} \circ P_{\mathrm{CE}} \circ P_{\mathrm{OC}} \circ R_{\mathrm{OAEC}}(v_4)$$

(4.28) $$v_6 = P_{\mathrm{AO}} \circ P_{\mathrm{DA}} \circ P_{\mathrm{GD}} \circ R_{\mathrm{GFBD}} \circ P_{\mathrm{DG}} \circ P_{\mathrm{AD}} \circ P_{\mathrm{OA}}(v_5)$$
$$= P_{\mathrm{AO}} \circ P_{\mathrm{DA}} \circ R_{\mathrm{DGFB}} \circ P_{\mathrm{AD}} \circ P_{\mathrm{OA}} \circ (v_5)$$
$$= R_{\mathrm{OBDA}} \circ P_{\mathrm{BO}} \circ R_{\mathrm{BDGF}} \circ P_{\mathrm{OB}} \circ R_{\mathrm{OADB}} \circ (v_5)$$

Now we calculate v_i ($i = 1, \dots, 6$) in order. It follows directly from Proposition 4.2 that

(4.29) $$\mathrm{v}_1 = \mathrm{v}_0 - (-1)^{\mathbf{pq}}\tilde{\Omega}(\gamma(\cdot, \cdot, 0))(v_0)\, d_1 d_2$$

The calculations of v_2 and v_3 are similar, so we present details of the former calculation, but simply note the result of the latter calculation, leaving the details to the reader:

(4.30) $$v_2 = v_1 - (-1)^{\mathbf{qr}}\tilde{\Omega}(\gamma(0, \cdot, \cdot))(v_1) d_2 d_3$$
[Proposition 4.2]
$$= v_0 - (-1)^{\mathbf{pq}}\tilde{\Omega}(\gamma(\cdot, \cdot, 0))(v_0) d_1 d_2 - (-1)^{\mathbf{qr}}\tilde{\Omega}(\gamma(0, \cdot, \cdot))(v_0 - (-1)^{\mathbf{pq}}\tilde{\Omega}(\gamma(\cdot, \cdot, 0)) \times (v_0) d_1 d_2) d_2 d_3 \qquad [(4.29)]$$
$$= v_0 - (-1)^{\mathbf{pq}}\tilde{\Omega}(\gamma(\cdot, \cdot, 0))(v_0) d_1 d_2 - (-1)^{\mathbf{qr}}\tilde{\Omega}(\gamma(0, \cdot, \cdot))(v_0) d_2 d_3$$

(4.31) $$v_3 = v_0 - (-1)^{\mathbf{pq}}\tilde{\Omega}(\gamma(\cdot, \cdot, 0))(v_0) d_1 d_2 - (-1)^{\mathbf{qr}}\tilde{\Omega}(\gamma(0, \cdot, \cdot))(v_0) d_2 d_3 + (-1)^{\mathbf{pr}}\tilde{\Omega}(\gamma(\cdot, 0, \cdot))(v_0) d_1 d_3$$

The three calculations of v_4, v_5 and v_6 are similar, so we present their details only in case of the first, leaving details of the other two calculation to the reader:

(4.32) $$v_4 = P_{\mathrm{AO}} \circ R_{\mathrm{AEGD}} \circ P_{\mathrm{OA}}(v_0 - (-1)^{\mathbf{pq}}\tilde{\Omega}(\gamma(\cdot, \cdot, 0))(v_0) d_1 d_2 - (-1)^{\mathbf{qr}}\tilde{\Omega}(\gamma(0, \cdot, \cdot))(v_0) d_2 d_3 + (-1)^{\mathbf{pr}}\tilde{\Omega}(\gamma(\cdot, 0, \cdot))(v_0) d_1 d_3 \quad [(4.31)]$$

$$
\begin{aligned}
&= P_{\mathrm{AO}} \circ R_{\mathrm{AEGD}} \circ (P_{\mathrm{OA}}(v_0) \\
&\quad - (-1)^{\mathbf{pq}} P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(0, \cdot, \cdot, 0))(v_0)) d_1 d_2 \\
&\quad - (-1)^{\mathbf{qr}} P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(0, \cdot, \cdot))(v_0)) d_2 d_3 \\
&\quad + (-1)^{\mathbf{pr}} P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(\cdot, 0, \cdot))(v_0)) d_1 d_3 \\
&= P_{\mathrm{AO}}(P_{\mathrm{OA}}(v_0) - (-1)^{\mathbf{pq}} P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(\cdot, \cdot, 0))(v_0)) d_1 d_2 \\
&\quad - (-1)^{\mathbf{qr}} P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(0, \cdot, \cdot))(v_0)) d_2 d_3 \\
&\quad + (-1)^{\mathbf{pr}} P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(\cdot, 0, \cdot))(v_0)) d_1 d_3 \\
&\quad + (-1)^{\mathbf{qr}} \tilde{\Omega}(\gamma(d_1, \cdot, \cdot))(P_{\mathrm{OA}}(v_0) \\
&\quad - (-1)^{\mathbf{pq}} P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(\cdot, \cdot, 0))(v_0)) d_1 d_2 \\
&\quad - (-1)^{\mathbf{qr}} P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(0, \cdot, \cdot))(v_0)) d_2 d_3 \\
&\quad + (-1)^{\mathbf{pr}} P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(\cdot, 0, \cdot))(v_0)) d_1 d_3) d_2 d_3 \\
&\quad \text{[Propositions 4.2 and 4.3]} \\
&= v_0 - (-1)^{\mathbf{pq}} \tilde{\Omega}(\gamma(\cdot, \cdot, 0))(v_0)) d_1 d_2 \\
&\quad - (-1)^{\mathbf{qr}} \tilde{\Omega}(\gamma(0, \cdot, \cdot))(v_0) d_2 d_3 \\
&\quad + (-1)^{\mathbf{pr}} \tilde{\Omega}(\gamma(\cdot, 0, \cdot))(v_0) d_1 d_3 \\
&\quad + (-1)^{\mathbf{qr}} P_{\mathrm{AO}}(\tilde{\Omega}(\gamma(d_1, \cdot, \cdot))(P_{\mathrm{OA}}(v_0))) d_2 d_3
\end{aligned}
$$

$$
\begin{aligned}
v_5 &= v_0 - (-1)^{\mathbf{pq}} \tilde{\Omega}(\gamma(\cdot, \cdot, 0))(v_0) d_1 d_2 \\
&\quad - (-1)^{\mathbf{qr}} \tilde{\Omega}(\gamma(0, \cdot, \cdot,))(v_0) d_2 d_3 \\
&\quad + (-1)^{\mathbf{pr}} \tilde{\Omega}(\gamma(\cdot, 0, \cdot))(v_0) d_1 d_3 \\
&\quad + (-1)^{\mathbf{qr}} P_{\mathrm{AO}}(\tilde{\Omega}(\gamma(d_1, \cdot, \cdot))(P_{\mathrm{OA}}(v_0))) d_2 d_3 \\
&\quad + (-1)^{\mathbf{pq}} P_{\mathrm{CO}}(\tilde{\Omega}(\gamma(\cdot, \cdot, d_3))(P_{\mathrm{OC}}(v_0)) d_1 d_2
\end{aligned} \tag{4.33}
$$

$$
\begin{aligned}
\mathrm{v}_6 &= v_0 - (-1)^{\mathbf{pq}} \tilde{\Omega}(\gamma(\cdot, \cdot, 0))(v_0) d_1 d_2 \\
&\quad - (-1)^{\mathbf{qr}} \tilde{\Omega}(\gamma(0, \cdot, \cdot))(v_0) d_2 d_3 \\
&\quad + (-1)^{\mathbf{pr}} \tilde{\Omega}(\gamma(\cdot, 0, \cdot))(v_0) d_1 d_3 \\
&\quad + (-1)^{\mathbf{qr}} P_{\mathrm{AO}}(\tilde{\Omega}(\gamma(d_1, \cdot, \cdot))(P_{\mathrm{OA}}(v_0))) d_2 d_3 \\
&\quad + (-1)^{\mathbf{pq}} P_{\mathrm{CO}}(\tilde{\Omega}(\gamma(\cdot, \cdot, d_3))(P_{\mathrm{OC}}(v_0))) d_1 d_2 \\
&\quad - (-1)^{\mathbf{pr}} P_{\mathrm{BO}}(\tilde{\Omega}(\gamma(\cdot, d_2, \cdot))(P_{\mathrm{OB}}(v_0))) d_1 d_3
\end{aligned} \tag{4.34}
$$

It should be the case by Theorem 4.4 that $v_6 = v_0$. Therefore

$$(4.35)\quad \begin{aligned}&(-1)^{\mathbf{pq}}\tilde{\Omega}(\gamma(\cdot, \cdot, 0))(v_0)d_1d_2\\ &+ (-1)^{\mathbf{qr}}\tilde{\Omega}(\gamma(0, \cdot, \cdot))(v_0)d_2d_3\\ &- (-1)^{\mathbf{pr}}\tilde{\Omega}(\gamma(\cdot, 0, \cdot))(v_0)d_1d_3\\ &- (-1)^{\mathbf{qr}}P_{\mathrm{AO}}(\tilde{\Omega}(\gamma(d_1, \cdot, \cdot))(P_{\mathrm{OA}}(v_0)))d_2d_3\\ &- (-1)^{\mathbf{pq}}P_{\mathrm{OC}}(\tilde{\Omega}(\gamma(\cdot, \cdot, d_3))(P_{\mathrm{OC}}(v_0)))d_1d_2\\ &+ (-1)^{\mathbf{pr}}P_{\mathrm{BO}}(\tilde{\Omega}(\gamma(\cdot, d_2, \cdot))(P_{\mathrm{OB}}(v_0))))d_1d_3 = 0\end{aligned}$$

By multiplying by $(-1)^{\mathbf{pq}+\mathbf{pr}+\mathbf{qr}}$ in (4.35), we have

$$(4.36)\quad \begin{aligned}&(-1)^{\mathbf{pr}+\mathbf{qr}}\tilde{\Omega}(\gamma(\cdot, \cdot, 0))(v_0)d_1d_2\\ &+ (-1)^{\mathbf{pq}+\mathbf{pr}}\tilde{\Omega}(\gamma(0, \cdot, \cdot))(v_0)d_2d_3\\ &- (-1)^{\mathbf{pq}+\mathbf{qr}}\tilde{\Omega}(\gamma(\cdot, 0, \cdot))(v_0)d_1d_3\\ &- (-1)^{\mathbf{pq}+\mathbf{pr}}P_{\mathrm{AO}}(\tilde{\Omega}(\gamma(d_1, \cdot, \cdot))(P_{\mathrm{OA}}(v_0)))d_2d_3\\ &- (-1)^{\mathbf{pr}+\mathbf{qr}}P_{\mathrm{OC}}(\tilde{\Omega}(\gamma(\cdot, \cdot, d_3))(P_{\mathrm{OC}}(v_0)))d_1d_2\\ &+ (-1)^{\mathbf{pq}+\mathbf{qr}}P_{\mathrm{BO}}(\tilde{\Omega}(\gamma(\cdot, d_2, \cdot))(P_{\mathrm{OB}}(v_0))))d_1d_3 = 0\end{aligned}$$

Since $v_0 \in E_{\gamma(0,0,0)}$ was chosen arbitrarily, the proof is complete. ■

We conclude this section by discussing curvatures of the second kind of the induced superconnections dealt with in Section 3. Let $\eta: F \to M$ be another supervector bundle over the same base space embellished with a superconnection ∇', as in that section.

Proposition 4.6. For any $\gamma \in M^{(\mathbf{0},\mathbf{1})^2}$ we have

$$(4.37)\qquad \tilde{\Omega}_{\zeta \oplus \eta}(\gamma) = \tilde{\Omega}_{\zeta}(\gamma) \oplus \tilde{\Omega}_{\eta}(\gamma)$$

where $\tilde{\Omega}_{\zeta \oplus \eta}$, $\tilde{\Omega}_{\zeta}$, and $\tilde{\Omega}_{\eta}$ denote the curvature forms of the second kind of superconnections $\nabla \oplus \nabla'$, ∇, and ∇', respectively.

Proof. Let $v \oplus v' \in (E \oplus F_{\gamma(0,0)})$. We assume that $\gamma \in M^{D(\mathbf{p})\times D(\mathbf{q})}$. Let $d_1 \in D(\mathbf{p})$ and $d_2 \in D(\mathbf{q})$. Let $t_1 = \gamma(\cdot, 0)$, $t_2 = \gamma(d_1, \cdot)$, $t_3 = \gamma(0, \cdot)$, and $t_4 = \gamma(\cdot, d_2)$. By Proposition 2.2, we have

$$(4.38)\quad \begin{aligned}&(-1)^{\mathbf{pq}}\tilde{\Omega}_{\zeta\oplus\eta}(\gamma)(v \oplus v')d_1d_2\\ &= (v \oplus v')\end{aligned}$$

$$- \ q^{\nabla\oplus\nabla'}_{(t_3,d_2)} \circ q^{\nabla\oplus\nabla'}_{(t_4,d_1)} \circ p^{\nabla\oplus\nabla'}_{(t_2,d_2)} \circ p^{\nabla\oplus\nabla'}_{(t_1,d_1)}(v \oplus v')$$

$$= \{v - q^{\nabla}_{(t_3,d_2)} \circ q^{\nabla}_{(t_4,d_1)} \circ p^{\nabla}_{(t_2,d_2)} \circ p^{\nabla}_{(t_1,d_1)}(v)\}$$

$$\oplus \ \{v' - q^{\nabla'}_{(t_3,d_2)} \circ q^{\nabla'}_{(t_4,d_1)} \circ p^{\nabla'}_{(t_2,d_2)} \circ p^{\nabla'}_{(t_1,d_1)}(v')\}$$

$$= (-1)^{\mathbf{pq}}\tilde{\Omega}_\zeta(\gamma)(v)d_1d_2 \oplus (-1)^{\mathbf{pq}}\tilde{\Omega}_\eta(\gamma')(v')d_1d_2$$

$$= (-1)^{\mathbf{pq}}\{\tilde{\Omega}_\zeta(\gamma)(v) \oplus \tilde{\Omega}_\eta(\gamma')v')\}d_1d_2$$

Therefore the desired proposition obtains. ■

Proposition 4.7. Let $\gamma \in M^{D(\mathbf{p})\times D(\mathbf{p})}$ and $\hat{v} \in \mathscr{L}(\zeta, \eta)_{\gamma(0,0)}$. Then we have

(4.39) $$\hat{\tilde{\Omega}}(\gamma)(\hat{v})$$
$$= (\tilde{\Omega}_\eta(\gamma)) \circ \hat{v}_{\mathbf{e}} + (-1)^{\mathbf{p}+\mathbf{q}}(\tilde{\Omega}_\eta(\gamma)) \circ \hat{v}_{\mathbf{o}} - \hat{v} \circ (\tilde{\Omega}_\zeta(\gamma))$$

where $\hat{\tilde{\Omega}}$, $\tilde{\Omega}_\zeta$ and $\tilde{\Omega}_\eta$ denote the curvature forms of the second kind of superconnections $\hat{\nabla}$, ∇ and ∇' respectively.

Proof. Let $d_1 \in D(\mathbf{p})$ and $d_2 \in D(\mathbf{q})$. By Proposition 4.2 we have

(4.40) $$(-1)^{\mathbf{pq}}\hat{\tilde{\Omega}}(\gamma)(\hat{v})d_1d_2$$
$$= \hat{v} - q^{\hat{\nabla}}_{(t_3,d_2)} \circ q^{\hat{\nabla}}_{(t_4,d_1)} \circ p^{\hat{\nabla}}_{(t_2,d_2)} \circ p^{\hat{\nabla}}_{(t_1,d_1)}(\hat{v})$$
$$= \hat{v} - q^{\nabla}_{(t_3,d_2} \circ q^{\nabla}_{(t_4,d_1)} \circ p^{\nabla}_{(t_2,d_2)} \circ p^{\nabla}_{(t_1,d_1)} \circ \hat{v}$$
$$\circ \ q^{\nabla}_{(t_1,d_1)} \circ q^{\nabla}_{(t_2,d_2)} \circ p^{\nabla}_{(t_4,d_1)} \circ p^{\nabla}_{(t_3,d_2)})$$
$$= \hat{v} - (\mathrm{id}_{F_{\gamma(0,0)}} - (-1)^{\mathbf{pq}}\tilde{\Omega}_\eta(\gamma)d_1d_2) \circ \hat{v} \circ (\mathrm{id}_{E_{\gamma(0.0)}}$$
$$+ \ (-1)^{\mathbf{pq}}\tilde{\Omega}_\zeta(\gamma)d_1d_2)$$
$$= \hat{v} - (\mathrm{id}_{F_{\gamma(0,0)}} - (-1)^{\mathbf{pq}}d_1d_2(\tilde{\Omega}_\eta(\gamma))_{\mathbf{e}}$$
$$- \ (-1)^{\mathbf{pq}+\mathbf{p}+\mathbf{q}}d_1d_2(\Omega_\eta(\gamma))_{\mathbf{o}}) \circ \hat{v} \circ (\mathrm{id}_{E_{\gamma(0,0)}}$$
$$+ \ (-1)^{\mathbf{pq}}\tilde{\Omega}_\zeta(\gamma)d_1d_2)$$
$$= (-1)^{\mathbf{pq}}(\tilde{\Omega}_\eta(\gamma))_{\mathbf{e}} \circ \hat{v}_{\mathbf{e}}d_1d_2$$
$$+ \ (-1)^{\mathbf{pq}+\mathbf{p}+\mathbf{q}}(\tilde{\Omega}_\eta(\gamma))_{\mathbf{e}} \circ \hat{v}_{\mathbf{o}}d_1d_2$$
$$+ \ (-1)^{\mathbf{pq}}(\tilde{\Omega}_\eta(\gamma))_{\mathbf{o}} \circ \hat{v}_{\mathbf{e}}d_1d_2$$
$$+ \ (-1)^{\mathbf{pq}+\mathbf{p}+\mathbf{q}}(\tilde{\Omega}_\eta(\gamma))_{\mathbf{o}} \circ \hat{v}_{\mathbf{o}}d_1d_2 - (-1)^{\mathbf{pq}}\hat{v} \circ (\Omega_\zeta(\gamma))d_1d_2$$
$$= (-1)^{\mathbf{pq}}(\tilde{\Omega}_\eta(\gamma)) \circ \hat{v}_{\mathbf{e}}d_1d_2 + (-1)^{\mathbf{pq}+\mathbf{p}+\mathbf{q}}(\tilde{\Omega}_\eta(\gamma)) \circ \hat{v}_{\mathbf{o}}d_1d_2$$

$$
\begin{aligned}
& - (-1)^{\mathbf{pq}} \hat{v} \circ (\tilde{\Omega}_{\zeta}(\gamma)) d_1 d_2 \\
= & (-1)^{\mathbf{pq}} \{ (\tilde{\Omega}_{\eta}(\gamma)) \circ \hat{v}_{\mathbf{e}} + (-1)^{\mathbf{p}+\mathbf{q}} (\tilde{\Omega}_{\eta}(\gamma)) \circ \hat{v}_{\mathbf{o}} \\
& - \hat{v} \circ (\tilde{\Omega}_{\zeta}(\gamma)) \} d_1 d_2
\end{aligned}
$$

Therefore the desired proposition obtains. ■

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